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MEASURE-PRESERVING RANK ONE TRANSFORMATIONS

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Abstract. Rank 1 transformations serve as a source of examples in ergodic theory showing a variety of algebraic, asymptotic, and spectral properties of dynamical systems. The properties of a rank one transformation are closely related to the structure of the semigroup of weak limits of its powers. In this vein, known and new constructions of transformations are studied.

1. Introduction

In this paper, we use the term *transformation* to mean an invertible measure-preserving transformation of the Lebesgue space $(X, \mathcal{B}, \mu)$. So far we are speaking of the standard probability space, but later we also consider transformations of a space of infinite measure. The transformations form a group, Aut, which is naturally equipped with a complete metric. Rokhlin and Halmos proved that weak but not strong mixing is Baire generic. The fact that periodic transformations are dense in Aut has led to the method of approximation by periodic transformations developed by Katok, Oseledets, Stepin, and others for discovering generic properties of transformations and constructing examples. A rank one transformation can be approximated in the sense of [4], but its definition does not involve external approximating transformations.

A transformation T is said to be of *rank one* if some sequence of measurable partitions

$$\xi_j = \{E_j, TE_j, T^2E_j, \dots, T^{h_j-1}E_j, \tilde{E}_j\}$$

of the phase space tends to the partition into points. This means that every set $A \in \mathcal{B}$ can be approximated by some sequence ξ_j of measurable sets A_j,

$$\mu(A\Delta A_j) \to 0, \quad j \to \infty.$$

The sequence of partitions can be modified so that the resulting new sequence is monotone [30]. This leads to a different definition of rank one in which the transformation is a construction uniquely determined by the given parameters.

Construction of a rank one transformation. Let there be given a sequence of integer vectors

$$\bar{s}_j = (s_j(1), s_j(2), \dots, s_j(r_j - 1), s_j(r_j)), \quad r_j > 1.$$

Set $h_1 = 1$. We extend the definition of the phase space and the transformation by induction. At any step, what has been defined at the previous steps is never changed.

At stage j, one has a partially defined transformation T that is a usual permutation of disjoint intervals forming the tower

$$E_j, TE_j, T^2E_j, \dots, T^{h_j-1}E_j.$$

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Let us cut the interval E_j into r_j intervals $E_j^1, E_j^2, E_j^3, \dots, E_j^{r_j}$ of equal measure. Consider the columns

$$E_j^i, TE_j^i, T^2E_j^i, \dots, T^{h_j-1}E_j^i, \quad i = 1, 2, \dots, r_j.$$

We add $s_j(i)$ new intervals over the ith column and obtain the set of intervals

$$E_j^i, TE_j^i, T^2E_j^i, \dots, T^{h_j+s_j(i)-1}E_j^i.$$

(The intervals are disjoint and have the same measure.) Let

$$T^{h_j+s_j(i)}E_j^i = E_j^{i+1}$$

for all $i < r_j$. Thus, we have combined the columns into the stage $j+1$ tower

$$E_{j+1}, TE_{j+1}, T^2E_{j+1}, \dots, T^{h_{j+1}}E_{j+1},$$

where

$$E_{j+1} = E_j^1, \quad h_{j+1} = h_j r_j + \sum_{i=1}^{r_j} s_j(i).$$

Continuing the construction, we obtain a measure-preserving transformation T on the union X of all intervals. If the measure of X is finite, then we normalize it.

A wider class of constructions in which the intervals may change their length under the action of T can be defined in a similar way. The paper [58] presents a transformation of this kind with a quasi-invariant measure that is not equivalent to any invariant measure.

The following examples are well known in ergodic theory:

(1) The Chacon transformations [33, 34]

$$\bar{s}_j = (2, 3, 1), \quad \bar{s}_j = (0, 1).$$

(2) The Ornstein stochastic constructions [59]

$$s_j(i) = b_j + a_j(i) - a_j(i+1), \quad 1 \le a_j(i) \le b_j \to \infty.$$

(3) The del Junco–Rudolph transformation [49]

$$\bar{s}_j = s(0, 0, \dots, 0, 0, 1, 0, 0, \dots, 0, 0), \quad r_j \to \infty.$$

(4) The Katok construction [51]

$$\bar{s}_j = (0, 0, \dots, 0, 0, 1, 1, \dots, 1, 1), \quad r_j \to \infty.$$

We in particular consider the following properties of transformations: mixing, multiple weak mixing, triviality of the centralizer, absence of factors, rigidity, minimal self-joining, simplicity of self-joining, disjointness of convolution powers of the spectral measure, and simplicity of the spectrum of symmetric powers.

The properties of constructions are completely determined by the sequence of spacers $\bar{s}_j$ that specify the roof over the tower. For this reason, one can draw an analogy between the construction of rank one transformations and a method described in fiction:

> There was a most ingenious Architect who had contrived a new Method for building Houses, by beginning at the Roof, and working downwards to the Foundation…
>
> *Jonathan Swift, Gulliver's Travels, Pt. III, Ch. 5*

2. Survey of results

Mixing constructions. A transformation T of a probability space is said to have the mixing property if

$$\mu(T^i A \cap B) \to \mu(A)\mu(B) \quad \text{as} \quad i \to \infty$$

for any $A, B \in \mathcal{B}$. The class of rank one transformations specified by a random parameter was introduced by Ornstein [59], who proved that mixing transformations with a trivial centralizer form a parametric majority in this class. Mixing transformations of rank one have multiple mixing [52, 10].

A new approach to studying the spectra of rank one transformations was proposed in [31], where the singularity of the spectrum was established for a parametric majority of Ornstein transformations. A general result on the singularity of the spectrum of transformation constructions with a sequence r_j of tempered growth is given in [57]. The mixing property for a class of staircase constructions was proved in [26]. Slight modifications of staircase constructions were applied to the study of multiplicities of the spectrum in [13].

Minimal self-joinings. Rudolph [61] introduced a rank one transformation with minimal self-joinings (MSJ). This property guarantees the triviality of the centralizer and the absence of factors (i.e., there exist no proper invariant σ-algebras). Further, this property implies the minimality of the centralizer and of the structure of factors for all Cartesian powers of the transformation. Informally speaking, minimality means that the only objects present are those that necessarily exist. The transformation T with minimal self-joinings was used to construct a variety of counterexamples, a collection of which can be found in [61]. Consider the products

$$T' = T \times T \times T \times \dots, \quad T'' = T \odot T \times T \times \dots,$$

where the factor $T \odot T$ of the product $T \times T$ is the restriction of $T \times T$ to the algebra of fixed sets of the symmetry $(x, y) \to (y, x)$. Then T' and T'' are weakly isomorphic in the sense of Sinai; i.e., T' is isomorphic to a factor of T'' and vice versa. However, they are not isomorphic, which is a consequence of the MSJ property. Another example is as follows: the product $T \times T \times T$ has a root of order 3 but no roots of other orders.

A *self-joining of order* 2 of the transformation T is a measure ν on $X \times X$ such that $(T \times T)\nu = \nu$ and $\nu(X \times A) = \nu(A \times X) = \mu(A)$ for any $A \in \mathcal{B}$.

Associated with a transformation S commuting with T is the self-joining Δ_S on $(X \times X)$ defined by $\Delta_S = (Id \times S)\Delta$, where Δ is the diagonal measure, $\Delta(A \times B) = \mu(A \cap B)$.

A transformation that has no ergodic self-joinings other than Δ_{T^n} and $\mu \times \mu$ is said to have the property of minimal self-joinings of order two denoted by MSJ(2). The related notion Simpl(2) of simplicity of joinings means that there exist no ergodic self-joinings other than Δ_S and $\mu \times \mu$.

Minimal self-joinings of higher orders are defined in a similar way. The informal definition of MSJ is that the transformation T only allows obvious self-joinings of all orders. For an introduction to this topic, e.g., see [55, 42, 63, 11].

Generic properties. Generic properties of transformations in Aut are properties of some rank one transformations. Relatively recently, the categorical genericity of the MSJ property has been discovered. Tikhonov [25] rediscovered Alpern's metric [29] on the space Mix of mixing transformations, proving the completeness of Mix and establishing the genericity of properties such as the singularity of the spectrum and multiple mixing. Bashtanov [2] discovered the genericity of rank one in the space Mix and, taking into account the results in the papers [52, 10, 54], the genericity of the MSJ property.

In the group Aut, generic transformations have roots [56], and they are group extensions [1]. (It is not known whether a generic transformation is a relatively weakly mixing extension [15].) A generic transformation can be included in a flow [62], the inclusion being nonunique; the centralizer of a generic transformation contains infinite-dimensional tori [24]. None of this is true for the space Mix, where a trivial centralizer and the absence of factors are generic.

Nonmixing constructions. Constructions with spacers of the form $\bar{s}_j = (2,3,1)$ and $\bar{s}_j = (0,1)$ were considered in [33, 34] as examples of nonmixing rank one transformations without roots and with continuous spectrum. The transformation with $\bar{s}_j = (0,1)$ is called the classical Chacon transformation; later on, we discuss its properties in detail. For the modified Chacon transformation (the spacers $\bar{s}_j = (0,1,0)$), the MSJ property [48] and the disjointness of convolution powers of the spectral measure [60] were established, and the structure of the weak closure was described [45].

A natural generalization of Chacon transformations is given by *bounded constructions* (i.e., rank one constructions all of whose parameters are bounded). It was proved in [69] that all nonrigid weakly mixing bounded constructions have minimal self-joinings, and the pairwise disjointness of all positive powers of weakly mixing bounded constructions was established in connection with the paper [32]. A proof of the spectral disjointness of powers of weakly mixing bounded constructions was obtained in [40].

There exists a class of nonmixing transformations formed by *semibounded constructions* (where some of the parameters are bounded). The simplicity of spectrum of the powers $\widehat{T}^{\odot n}$ is easier to establish for many semibounded constructions than for bounded ones. A semibounded rank one transformation T such that the tensor products $\widehat{T} \otimes \widehat{T}^m$ have simple singular spectrum for $1 < m < 2020$ and the spectrum of the products $\widehat{T} \otimes \widehat{T}^n$ for $n \geq 2020$ is countably multiple and Lebesgue was constructed in [22]. The semibounded self-similar rank one transformation considered in [70] induces a Gaussian automorphism G with singular spectrum such that the power G^3 is isomorphic to the tensor power $G \otimes G \otimes G$. For this automorphism G, the set of spectral multiplicities of the power G^{3^p} is $\{3^p, \infty\}$. But then the set of spectral multiplicities of elements of the Gaussian flow G_t is not constant.

Weak closure of a rank one action. The following application of the weak limits of powers of a transformation is well known in ergodic theory. In connection with Kolmogorov's problem on the group property of the spectrum of a transformation, Oseledets and Stepin (see [8, 23]) considered weak limits of the form $aI + (1-a)\Theta$, where Θ is the orthogonal projection onto the space of constants in L_2.

The weak limits $aI + (1-a)\widehat{T}$ were used in the theory of joinings [63, 11] and were later applied in spectral theory of dynamical systems (see [39]). In the case of mixing, which means the convergence $\widehat{T}^i \to \Theta$, where $\widehat{T}$ is the operator induced by the transformation T, there exist no nontrivial weak limits. However, the weak limit technique can be used even the case of mixing [64, 16]. In terms of nonstandard analysis, this is explained by the fact that the weak closure of powers of a mixing transformation contains limits that are infinitely close to the operator Θ and force the desired spectral property to hold.

For a probability space, proving the mixing property of a rank one transformation is a nontrivial problem; it requires controlling the behavior of the quantities $\mu(T^i A \cap B)$ for all large i. Describing the semigroup of all weak limits of powers of a transformation is a more general and difficult problem in the class of weakly mixing actions; see [45, 69, 65]. If the weak closure of powers of a weakly mixing rank one transformation T is polynomial, that is, consists of linear combinations of powers of $\widehat{T}$, then T has the MSJ property.

The problem of describing the weak closure of a rank one action can be much simpler in the case of infinite measure of the phase space. For example, the mixing property ($\widehat{T}^i \to_w 0$) and the rapid decay of Fourier coefficients of the spectral measure for Sidon constructions follow from the definition of these constructions [18]. It is relatively easy to describe the nontrivial semigroups of some semibounded nonmixing constructions (see [22, 5]).

The notion of rank one transformation has generalizations such as finite rank, local rank (see [12]), and group actions of rank one. The use of new invariants expands the range of questions and researchers' opportunities. Del Junco [50] used a rank one group action with minimal self-joinings to find a simple transformation without minimal factors. His method has been developed and found a number of new applications [38]. Ageev used a similar technique as well; he considered a generic action of a specially selected group whose subaction has the desired spectral property. This has led to the discovery of ergodic transformations with a given multiplicity of the homogeneous spectrum [27].

Main content of the article. The paper studies the relations of the weak closure of an action to its spectral properties and the structure of its self-joinings. This line of research is most relevant for rank one transformations. With increasing rank, the relations become weaker or disappear altogether. The outline of the remaining part of the article is as follows.

Section 3: Weak closure of the action and the Markov centralizer. If an indecomposable Markov operator commutes with a rank one transformation T, then, informally speaking, it can be approximated in the weak topology by parts of two powers of the transformation.

Section 4: Bounded constructions. The simplest bounded construction with continuous spectrum, the classical Chacon transformation, is considered. A number of its properties is proved.

Section 5: Semibounded constructions. Modifications of the Chacon transformation display the latent rigidity phenomenon and the presence of all polynomial limits. The stochastic Chacon transformation is discussed.

Section 6: Stochastic constructions. A method for proving the mixing property of Ornstein transformations is explained.

Section 7: Genericity of rank one, nonstandard mixing, and nongeneric transformations. It is shown that the conjugacy class of the Ornstein transformation T is everywhere dense in the space Mix. In this case, the idea of a random coboundary is used twice: it is embedded in the Ornstein construction T itself and is applied virtually when choosing a suitable conjugation. Invariants in the class of mixing constructions are discussed. A sufficient condition is given for a compact set not to meet conjugacy classes containing a dense G_δ set.

Section 8: Algebraic spacers instead of random ones. Galois fields are a source of quasi-random sequences and hence permit one to derandomize stochastic constructions. The properties of constructions are similar to Ornstein's, but now algebra replaces statistics.

Section 9: Staircase constructions. The technique of proving the mixing property for staircase constructions is explained.

Section 10: An explicit mixing construction with double spectrum. It is well known that there exist mixing staircase constructions T such that $T \times T$ has a homogeneous spectrum of multiplicity 2. However, a specific example has not yet been indicated. We estimate the growth rate of the parameters r_j needed for the desired effect to take place. For example, one can take $r_j = [\ln(j + 8)]$.

Section 11: Infinite transformations and self-similar constructions. Simple infinite constructions with unusual properties, for example, with self-similarity of the spectrum, are proposed. Such constructions are of interest as applications to Poisson and Gaussian actions. A simple example of a transformation that is not conjugate to the inverse of itself is given. Sidon constructions whose Cartesian square is dissipative are discussed.

Section 12: Concluding remarks as well as general and special questions about rank one transformations finish the article.

3. Weak closure of the action and the Markov centralizer

Centralizer and factors. King's theorem [56] states that if a transformation S commutes with a rank one transformation T, then $\widehat{T}^{k(j)} \to \widehat{S}$ for some sequence $k(j)$. (Here we speak of the weak convergence, which in this case coincides with the strong operator convergence.) As a corollary, it follows that if S is not a power of T, then T is rigid, because $\widehat{T}^{k(j+1)-k(j)} \to I$. Any proper factor of a rank one transformation is rigid [56]. The theorem has a number of applications (see [41, 44]). This raises the general question as to what the relation between the weak closure of an action and a Markov operator commuting with it is.

Self-joinings and the Markov centralizer. We know that for a rank one transformation T and a self-joining of the form $\nu = \Delta_S$ there exists a sequence $k(j)$ such that $\Delta^{k(j)} \to \nu$. Only a partial approximation has been proved in the general case:

If ν is an ergodic self-joining of order 2 of a rank one transformation T, then $\Delta^{k(j)} \to \frac{1}{2}\nu + \dots$ for some sequence $k(j)$.

Corollary. *Rank one mixing transformations have the property* MSJ(2).

The statement remains valid in the case of infinite measure spaces; see [22]. Thus, in the case of a probability space, the minimal self-joining property of $\mathbb{Z}$-actions of rank one is equivalent to their weak closure being contained in a convex combination of elements of the action and the orthogonal projection of Θ onto the space of constants in $L_2(X, \mathcal{B}, \mu)$. The scarce structure of the weak closure leads to the scarce structure of self-joinings with all the ensuing consequences. The presence of nontrivial polynomial limits results in a number of spectral effects.

An operator P in $L_2(X, \mu)$ is called a *Markov operator* if P is positive (i.e., takes nonnegative functions to nonnegative ones) and the operators P and P^* preserve the integral. Associated with self-joinings of a transformation T are Markov operators commuting with $\widehat{T}$. This correspondence is given by the formula

$$(Pf, g) = \int_{X\times X} f \otimes g d\nu.$$

In the operator language, the above-mentioned assertion can be stated as follows:

$$T^{k(j)} \to \frac{1}{2}P + P',$$

where the Markov operator P corresponds to an ergodic self-joining ν and P' is a positive operator (which may be zero in the case of an infinite space). Here positivity means that $P'f \geq 0$ for $f \geq 0$. The operator P is indecomposable: it cannot be represented as the half-sum of distinct Markov operators commuting with $\widehat{T}$. Associated with an ergodic self-joining of a transformation is an operator that is an extreme point in its Markov centralizer.

The following assertion (where the case of a probability space is considered) shows that an ergodic self-joining can be approximated by parts of the measures $\Delta^{k(j)}$.

Let ν be an ergodic self-joining of a rank one transformation T. There exists a sequence k_j such that

$$\nu(A \times B) = \lim_j \mu(T^{k_j} A \cap B \;|Y_j)$$

for any $A, B \in \mathcal{B}$, where Y_j is part of the stage j tower and the measure of Y_j is not less than $\frac{1}{2}$.

It can be seen from the proofs [10] that the self-joining is approximated by two parts,

$$\nu(A \times B) = \lim_j \big(\mu(T^{k_j} A \cap B \cap Y_j) + \mu(T^{k'_j} A \cap B \cap Y'_j)\big),$$

where Y_j and Y'_j are disjoint parts of the stage j tower and the sum of their measures tends to 1. We give an equivalent operator statement.

Theorem 3.1. *Let T be a rank one transformation, and let P be an indecomposable Markov operator commuting with $\widehat{T}$. Then there exist sequences k_j and k'_j and a sequence of sets $Y_j \in \mathcal{B}$ such that*

$$\widehat{Y}_j \widehat{T}^{k_j} + (I - \widehat{Y}_j)\widehat{T}^{k'_j} \to_w \; P,$$

where $\widehat{Y}_j$ is the operator of multiplication by the indicator function of the set Y_j.

Proof. Set

$$Y_j^k = \bigsqcup_{i=0}^{h_j - k} T^{i+k} E_j, \quad a_j^k = \frac{\nu(T^k E_j \times E_j)}{\mu(E_j)}, \quad 0 \le k < h_j,$$

$$Y_j^k = \bigsqcup_{i=0}^{h_j + k} T^{i} E_j, \quad a_j^k = \frac{\nu(E_j \times T^{-k} E_j)}{\mu(E_j)}, \quad -h_j < k < 0.$$

One can verify that

$$\nu(A \times B) = \nu_j(A \times B) := \sum_k a_j^k \mu(T^k A \cap B \cap Y_j^k)$$

for the self-joining ν of T and for ξ_j-measurable sets $A, B \in \mathcal{B}$.

Consider the normalized measures Δ_j^k given by the formula

$$\Delta_j^k(A \times B) = \mu(T^k A \cap B \,|Y_j^k)$$

for $|k| < (1-\delta)h_j$, where $\delta > 0$ is small.

If the self-joining ν is ergodic, then

$$\Delta_j^k(A \times B) \approx \nu(A \times B)$$

for a weighted majority (with respect to the weights a_j^k for a given j) of numbers k. This happens because if a convex sum of almost invariant normalized measures is close to a normalized ergodic measure, then most of these measures will be close to the ergodic measure. Otherwise, one could readily show that the measure ν is not an extreme point of the space of normalized invariant measures (see [10, 44]). The desired sequence $\Delta_j^{k_j} \to \nu$ can always be chosen to satisfy $\frac{|k_j|}{h_j} \to b \ge \frac{1}{2}$. This is possible owing to the projection properties of the measure ν, which imply that $\sum_k a_j^k > a > 0$ for $2|k| > 1 + \delta$. If k_j is taken to maximize b, then for $b < 1$ there exists an antipode k'_j such that $|k_j| + |k'_j| \approx h_j$ and $\Delta_j^{k'_j} \to \nu$. In the case of positive k_j, this gives the convergence

$$\nu(A \times B) = \lim_j \big(\mu\big(T^{k_j} A \cap B \cap Y_j^{k_j}\big) + \mu\big(T^{k'_j} A \cap B \cap Y_j^{h_j - k_j}\big)\big). \tag{3.1}$$

It remains to note that one can replace $Y_j^{h_j - k_j}$ with $X \setminus Y_j^{k_j}$ and restate (2.1) in operator terms. The proof of the theorem is complete. □

Flat roof case. If
$$\mu(T^{h_j}E_j|E_j) \to 1,$$
then one can take $k'_j = h_j - k_j$, and (2.1) becomes
$$\nu(A \times B) = \lim_j \mu(T^{k_j}A \cap B).$$
This result is known as King's theorem on the weak closure of powers of a flat roof transformation [44].

4. Bounded constructions, spectrum, and self-joinings

Consider the simplest rank one constructions with bounded parameters: the Kakutani–von Neumann transformation and the classical Chacon transformation.

Odometer. Fix a sequence of positive integers $r_j > 1$. Let us describe a construction with zero spacer vectors $\bar{s}_j = (0, 0, \dots, 0)$. The phase space X in this example is an interval of unit length. (It is convenient for the reader to think of it as a half-interval.) At stage $j = 1$, we have an interval E_1 and $h_1 = 1$.

Let the partially defined transformation S at stage j be a permutation of disjoint intervals $E_j, SE_j, S^2E_j, \dots, S^{h_j-1}E_j$. It is not defined yet on the last interval. Let us proceed to stage $j + 1$. Let us represent the interval E_j as the union of r_j intervals $E_j^1, E_j^2, E_j^3, \dots, E_j^{r_j}$ of the same length. For $i = 1, 2, \dots, r_j$, consider the columns E_j^i, SE_j^i, $S^2E_j^i, \dots, S^{h_j-1}E_j^i$ and set $S^{h_j}E_j^i = E_j^{i+1}$ for each $i < r_j$. We obtain the tower
$$E_{j+1}, SE_{j+1}, S^2E_{j+1}, \dots, S^{h_{j+1}-1}E_{j+1},$$
where $E_{j+1} = E_j^1$ and $h_{j+1} = h_j r_j$. The transformation S is only undefined on the last interval. Continuing this process infinitely, we define S on the entire X.

The transformation thus constructed is ergodic, because an invariant set A of positive measure has the same structure on all intervals and hence has full measure. This argument applies to all rank one transformations. Indeed, for large j some interval of the stage j tower mainly consists of elements of A, but in view of the invariance of A the same is true for all other intervals of the tower.

The odometer S induces the unitary operator $\widehat{S}$, $\widehat{S}f(x) = f(Sx)$, in the space $L_2(X, \mathcal{B}, \mu)$. The spectrum of $\widehat{S}$ is the group generated by the numbers $e^{2\pi mi/r_1r_2\dots r_n}$. The sequence $\widehat{S}^{h_j}$ strongly converges to the identity operator I. (Recall that such transformations are said to be rigid.) Note that all ergodic transformations with discrete spectrum are rigid and have rank one [47].

Isomorphism of powers. If, say, $r_j = 3$ for all j, then the corresponding odometer is isomorphic to the square of itself. Ageev [28] showed that there exist rank one transformations R with continuous spectrum that have a similar property. Explicit constructions are hardly easy to present, because their parameters are necessarily unbounded in view of the results in [69]. As a consequence of King's theorem [53], all transformations commuting with such an R have common conjugation with their squares. Such R cannot commute with a transformation of even period other than the identity transformation.

Induced transformation and self-joinings. The preprint [35] deals with the transformations $T = S_A$ induced by some odometer S on a specially selected subset $A \subset X$. Recall that $S_Ax = S^{n(x)}x$, where $x \in A$ and $n(x)$ is the time of first return of x to A. The authors obtained an example of a rigid rank one transformation without factors but with an extensive structure of self-joinings, which are relatively weakly mixing. Let us explain what this means. Let a Markov operator P correspond to a self-joining $\nu \neq \Delta$, and let a self-joining η correspond to the operator P^*P. If η is ergodic, then ν is called a relatively weakly mixing extension of the original system. Does a generic transformation always have such a self-joining? Even if it is given by a factor, i.e., under the additional condition $P^* = P = P^2$, the answer is unknown; see [15].

Classical Chacon transformation. Let some interval E_0 be given at the zero stage. At stage j, one has the tower

$$E_j, TE_j, T^2E_j, \dots, T^{h_j-1}E_j$$

consisting of h_j disjoint intervals of the same length. The transformation T is defined on all but the last interval as a normal transfer of intervals. At stage $j+1$, we represent E_j as the union $E_j = E_j^1 \sqcup E_j^2$ of disjoint intervals of the same length. We cut the stage j tower into two columns and add the spacer $T^{h_j}E_j^2$ above the second column,

$$\begin{aligned} &E_j^1, TE_j^1, T^2E_j^1, \dots, T^{h_j-1}E_j^1,\\ &E_j^2, TE_j^2, T^2E_j^2, \dots, T^{h_j-1}E_j^2, T^{h_j}E_j^2. \end{aligned}$$

By setting $T^{h_j}E_j^1 = E_j^2$ and $E_{j+1} = E_j^1$, we obtain the stage $j+1$ tower consisting of the $h_{j+1} = 2h_j + 1$ intervals

$$E_{j+1}, TE_{j+1}T^2E_{j+1}, \dots, T^{h_{j+1}-1}E_{j+1}.$$

The transformation T has not yet been defined on the last interval. Continuing the construction, we extend the definition of both the phase space X and the transformation T. Note that the sum of measures of all intervals added in the process is finite; the normalized Lebesgue measure on X is taken for the invariant probability measure.

Here is the list of properties of the classical Chacon transformation:

(i) It has the weak mixing property (there are no eigenfunctions except for constants) but not the strong mixing property.

(ii) The spectral measure σ_T of the transformation is mutually singular with the convolution $\sigma_T * \sigma_T$, and the product $\widehat{T} \otimes \widehat{T}$ has homogeneous spectrum of multiplicity 2.

(iii) The transformation T has the multiple weak mixing property (below we state the property WMix(2)): if $\widehat{T}^{m(i)}, \widehat{T}^{n(i)}, \widehat{T}^{m(i)-n(i)} \to_w \Theta$, then

$$\mu(A \cap \widehat{T}^{m(i)}B \cap \widehat{T}^{n(i)}C) \to \mu(A)\mu(B)\mu(C)$$

for any $A, B, C \in \mathcal{B}$.

(iv) The transformation T has the MSJ property; therefore, it has a trivial centralizer and no factors.

Let us show how one can establish these properties of the transformation.

(i) The absence of mixing follows from the existence of a weak limit: the sequence $\widehat{T}^{-h_j}$ weakly converges to the operator

$$P(\widehat{T}) = \sum_{k=1}^{\infty} 2^{-k}\widehat{T}^{k-1}.$$

Let us prove that the eigenfunctions of the operator $\widehat{T}$ are constants. Let $\widehat{T}f = \lambda f$; then $\widehat{T}^n f = \lambda^n f$. We obtain $|P(\lambda)f| = \alpha f$ for some λ with $|\lambda| = 1$. Since

$$|P(\lambda)| = \left|\sum_{k=1}^{\infty} 2^{-k}\lambda^{k-1}\right| = 1,$$

we see that $\lambda = 1$. But T is an ergodic transformation, and so f is constant.

(ii) The assertion that the spectral measure σ_T of the transformation is mutually singular with the convolution square $\sigma_T * \sigma_T$ is equivalent to saying that the restriction of the operator $\widehat{T}$ to the space $H = \mathrm{Const}^{\perp}$ has no nonzero intertwining with the product $\widehat{T} \otimes \widehat{T}$. Let us prove that there exists no nonzero intertwining.

Let $J\colon H \to H \otimes H$, and let the intertwining condition

$$J\widehat{T} = (\widehat{T} \otimes \widehat{T})J$$

be satisfied.

Let $\widehat{T}^{k_j} \to P = P(\widehat{T})$. Then

$$JP = (P \otimes P)J,\ (P \otimes P)^{-1}J = JP^{-1},$$

$$((I - a\widehat{T}) \otimes (I - a\widehat{T}))J = (1 - a)J(I - a\widehat{T}),$$

$$[(I \otimes I) + (\widehat{T} \otimes \widehat{T}) - (\widehat{T} \otimes I) - (I \otimes \widehat{T})]J = 0.$$

It follows from the relation

$$\sum_{i,j=0}^{n-1} (\widehat{T}^i \otimes \widehat{T}^j)[(I \otimes I) + (\widehat{T} \otimes \widehat{T}) - (\widehat{T} \otimes I) - (I \otimes \widehat{T})]J = 0$$

that

$$(I \otimes I)J + (\widehat{T}^n \otimes \widehat{T}^n)J - (\widehat{T}^n \otimes I)J - (I \otimes \widehat{T}^n)J = 0.$$

The weak mixing property of the transformation T is equivalent to the convergence $\widehat{T}^{n_i} \to_w 0$ (on the space H) for some sequence $n_i \to \infty$. It follows from the preceding that $(I \otimes I)J = 0$ and $J = 0$, as required.

Now let us give a proof of the stronger statement that $\widehat{T}\otimes\widehat{T}$ has homogeneous spectrum of multiplicity 2. This assertion also implies the above-mentioned disjointness, because if the measures σ_T and $\sigma_T * \sigma_T$ have a common component, then the maximum spectral multiplicity of $\widehat{T} \otimes \widehat{T}$ is at least 4.

Let f be a cyclic vector of the operator $\widehat{T}$, and let $a = \frac{1}{2}$. Consider the vectors

$$W_n = (I - a\widehat{T})^n f \otimes (I - a\widehat{T})^n f,\ n \in \mathbb{N}.$$

Let C_n be the cyclic space with cyclic vector W_n. The weak limits of the powers of the transformation form a semigroup. Therefore, they contain the operators P^m. Thus,

$$P^m(I - a\widehat{T})^n f \otimes P^m(I - a\widehat{T})^n f \in C_n$$

for all m. It follows that

$$(I - a\widehat{T})^k f \otimes P^m(I - a\widehat{T})^k f \in C_n$$

for $k = 0, 1, 2, \ldots, n$, but this leads to

$$\widehat{T}^k f \otimes f + \widehat{T}^k f \otimes f =: V_k \in C_n.$$

Since the vectors V_k and their shifts generate the entire space $L_2 \odot L_2$ of symmetric functions F $(F(x,y) = F(y,x))$, we have established that the cyclic spaces C_n approximate the entire space $L_2 \odot L_2$. Hence this space is cyclic. The restriction of the operator $\widehat{T}\otimes\widehat{T}$ to $L_2 \odot L_2$ has simple spectrum. The product of $\widehat{T} \otimes \widehat{T}$ by $L_2 \otimes L_2$ has homogeneous spectrum of multiplicity 2, because $\widehat{T} \otimes \widehat{T}$ is the direct sum of its restrictions to the spaces of symmetric and antisymmetric functions. These restrictions are isomorphic to each other, which readily follows from the spectral representation of the operator $\widehat{T} \otimes \widehat{T}$.

Conjecture: weakly mixing bounded constructions satisfy the spectrum minimality property (MS): the powers $\widehat{T}^{\odot n}$ have simple spectrum for all n.

(iii) Let us use the relation between multiple mixing and self-joining. Passing to a subsequence in i, denoting it again by i, we obtain

$$\mu(A \cap T^{m(i)}B \cap T^{n(i)}C) \to \nu(A \times B \times C)$$

for any $A, B, C \in \mathcal{B}$, where ν is a self-joining of order 3 whose projections onto the facets of the cube $X \times X \times X$ are equal to $\mu \times \mu$. Such a measure ν is uniquely related to the operator J by the formula

$$(J\chi_A, \chi_B \otimes \chi_C)_{L_2(\mu\times\mu)} = \nu(A \times B \times C);$$

further, the intertwining condition

$$JH \subset H \otimes H, \quad J\widehat{T} = (\widehat{T} \otimes \widehat{T})J$$

holds. By (ii), $JH = \{0\}$, and therefore,

$$JL_2 = \{\text{Const}\}, \quad \nu = \mu \times \mu \times \mu,$$

whence we obtain the desired convergence

$$\mu(A \cap T^{m(i)}B \cap T^{n(i)}C) \to \mu(A)\mu(B)\mu(C).$$

The result in [48] was generalized in [69].

Theorem 4.1. *All nonrigid completely ergodic bounded constructions have minimal self-joinings.*

The proof of this theorem is based on the delay effect. As a set passes through the spacers, some parts of it begin to lag behind the others. This trivializes the self-joining. For an ergodic self-joining $\nu \neq \Delta_{T^n}$, there exists an $s \neq 0$, $n(j) \to \infty$, and a sequence of sets C_j, $\mu(C_j) > c > 0$, such that simultaneously

$$\nu(A \times B) = \lim_j \mu(T^{n(j)}A \cap B \mid C_j) \quad \text{and} \quad \nu(A \times B) = \lim_j \mu(T^{n(j)-s}A \cap B \mid C_j).$$

Then

$$\nu(A \times B) = \nu(T^{-s}A \times B),$$

which, by the ergodicity of T, implies that $\nu = \mu \times \mu$. A similar argument was used in [48]. An algorithm for finding the sets C_j is described in [69].

(iv) Let us show how to prove MSJ for the construction $\bar{s}_j(0, 1)$. (The reasoning in the general case is similar.)

We rewrite (2.1) as

$$\nu(A \times B) = \lim_j \mu(T^{k_j}A \cap B \cap Y_j^{k_j}) + \dots$$

Case 1. Assume that

$$\frac{k_j}{h_j} \to 0, \quad k_j \to \infty, \quad \mu(Y_j^{k_j}) \to 1,$$

and we have

$$\nu(A \times B) = \lim_j \mu(T^{k_j}A \cap B).$$

Let us find $i(j)$ such that

$$h_{i(j)} \leq k_j < h_{i(j)+1} = 2h_{i(j)} + 1.$$

Consider the stage $i(j)$ tower and some floor F in the second column of this tower. Then the image $T^{k_j}F$ will consist of parts located on different floors in the tower: half of the image on one floor, a quarter on the floor below, etc. This is easy to understand if one imagines how the floor moves under the action of the transformation T when passing through the spacers above the second column. Thus, a part C_j of the phase space experiences a unit time delay, which leads to the relations

$$\nu(A \times B) = \lim_j \mu(T^{k_j}A \cap B) = \lim_j \mu(T^{k_j}A \cap B|C_j) = \nu(T^{-1}A \times B),$$

whence we obtain $\nu = \mu \times \mu$.

Case 2. Assume that $\frac{k_j}{h_j} > a > 0$ and

$$(1+4d)h_{i(j)} \le 4k_j < (2-4d)h_{i(j)}$$

for some small number $d > 0$ and infinitely many indices j. Let C_j be the union of the first $dh_{i(j)}$ floors. The measures of the sets C_j are greater than some positive constant. Without loss of generality, we assume that $\mu(C_j) \to c > 0$. Note that

$$T^{k_j}C_j,\, T^{2k_j}C_j \subset Y_j^{k_j},$$

but $\mu(T^{k_j}A \cap B | Y_j^{k_j}) \to \nu(A \times B)$, and so

$$\mu\big(T^{k_j}(A \cap C_j) \cap B\big) \to c\nu(A \times B) \tag{4.1}$$

and

$$\mu\big(T^{k_j}(A \cap T^{h_{i(j)-2}}C_j) \cap B\big) \to c\nu(A \times B). \tag{4.2}$$

However, simultaneously with the last convergence, one has

$$\mu\big(T^{k_j}(A \cap T^{h_{i(j)-2}}C_j) \cap B\big) \to c\nu(T^{-1}A \times B). \tag{4.3}$$

We need to prove (4.3). Let A_j and B_j denote the intersections of A and B, respectively, with the first columns of stages $i(j)-2$ and $i(j)-1$. Then

$$A = A_j \sqcup T^{h_{i(j)-2}}A_j \sqcup \dots, \quad B = B_j \sqcup T^{h_{i(j)-2}}B_j \sqcup T^{2h_{i(j)-2}+1}B_j \sqcup \dots$$

We have

$$\begin{aligned}\mu\big(T^{k_j}(A \cap T^{h_{i(j)-2}}C_j) \cap B\big) &= \mu\big(T^{k_j}(T^{h_{i(j)-2}}A_j \cap T^{h_{i(j)-2}}C_j) \cap T^{2h_{i(j)-2}+1}B_j\big)\\ &= \mu\big(T^{k_j}(T^{-1}A_j \cap T^{-1}C_j) \cap T^{h_{i(j)-2}}B_j\big) = \mu\big(T^{k_j}(T^{-1}A \cap T^{-1}C_j) \cap B\big).\end{aligned}$$

We substitute the set $T^{-1}A$ for A into (4.1), use the obvious convergence $\mu(C_j \Delta T^{-1}C_j) \to 0$, and establish (4.3),

$$\mu\big(T^{k_j}(T^{-1}A \cap T^{-1}C_j) \cap B\big) \to c\nu(T^{-1}A \times B).$$

In view of (4.2), we obtain

$$\nu(A \times B) = \nu(T^{-1}A \times B),\ \nu = \mu \times \mu.$$

If case 2 is not realized, then $h_{i(j)-1} \approx k_j$ (or $h_{i(j)-2} \approx k_j$). Note that this falls under the assumptions of case 1 with the new sequence $k_j' = k_j - h_{i(j)-1}$,

$$\mu(T^{k_j'}A \cap B) \to \nu(A \times B).$$

Thus, we have shown that only $\mu \times \mu$ and Δ_{T^k} (if $k_j = k$) can be ergodic joinings. This proves the MSJ property.

Disjointness of powers. The disjointness of positive powers of the transformation follows from MSJ: $\mu \times \mu$ is the only joining of the powers T^m and T^n for $m,n > 0$, $m \ne n$. An equivalent statement is that if $\widehat{T}^q J = J\widehat{T}^p$, $q \ne p$, where J is a Markov operator, then $J = \Theta$. The following statement was proved in [69] in connection with the paper [32].

Theorem 4.2. *The powers T^m and T^n with $0 < m < n$ are disjoint for all weakly mixing bounded constructions T.*

If a weakly mixing construction is not rigid, then it has the MSJ property. For a rigid weakly mixing bounded construction T, there exists an arbitrarily small number $\varepsilon > 0$ such that

$$\widehat{T}^{qh_j} \to_w Q(\widehat{T}) = q\varepsilon I + (1 - q\varepsilon)R(\widehat{T}) \quad \text{and} \quad \widehat{T}^{ph_j} \to_w P(\widehat{T}) = p\varepsilon I + (1 - p\varepsilon)R(\widehat{T}),$$

where the series $R(\widehat{T})$ satisfies $R(\widehat{T}) \neq I$ (see [69]). The disjointness of T^q and T^p for $p > q$ is obvious, because we obtain $R(\widehat{T})J = J$, which implies that $J = \Theta$.

Another convenient tool for proving the disjointness of powers of transformations is given by a lemma in [17].

Lemma. *Let S and T be completely ergodic transformations such that*

$$\widehat{S}^q J = J\widehat{T}^p,$$

where q and p are coprime and $J \neq \Theta$ is an indecomposable Markov operator. If polynomials Q and P satisfy

$$Q(\widehat{S})J = JP(\widehat{T}),$$

then there exists a polynomial R such that

$$Q(\widehat{S}) = R(\widehat{S}^q), \quad P(\widehat{T}) = R(\widehat{T}^p).$$

The result on the spectral disjointness of powers of bounded and some unbounded constructions was obtained in [40].

5. Semibounded constructions

Consider constructions with spacers of the form $\bar{s}_j = (0, s_j, 0)$. It is curious that if $s_j = [j^a]$, then the centralizer is trivial for $a = 0$ (the modified Chacon transformation), continual (as we will show) for $0 < a < 1$, and again trivial for $a \geq 1$.

We say that the sequence s_j grows slowly if it is monotone and takes each value r $N(r)$ times, where $N(r) \to \infty$ as $r \to \infty$. Examples are given by $s_j \sim rtj$ and $s_j \sim \ln j$.

Theorem 5.1. *Let T be a construction with spacers of the form $\bar{s}_j = (0, s_j, 0)$, where s_j is a slowly growing sequence. Then all series of the form*

$$P(\widehat{T}) = \sum_{k\in\mathbb{Z}} a_k \widehat{T}^k, \quad a_k \geq 0, \quad \sum_{k\in\mathbb{Z}} a_k = 1,$$

lie in the weak closure of powers of the operator $\widehat{T}$.

Corollary. *The transformation T is rigid; i.e., $\widehat{T}^{m_i} \to I$ for some sequence $m_i \to \infty$. The spectra of the symmetric tensor powers $\widehat{T}^{\odot n}$ are simple.*

Proof. It will be more convenient for us to write $h(j)$ instead of h_j. We denote the minimum number i such that $k = s_i$ by $j[k]$. Since $s_{j[k]+m} = k$ for $m = 1, 2, \ldots, N(k)-1$, we have

$$\widehat{T}^{h(j[k])} \approx_w \frac{1}{2}I + \frac{1}{2}\widehat{T}^{-k}$$

for large k. Now let us substitute h_j (or otherwise $-h_j$) for k. We obtain

$$\widehat{T}^{h(j[j[k]])} \approx_w \frac{1}{2}I + \frac{1}{4}I + \frac{1}{4}\widehat{T}^k,$$

$$\widehat{T}^{h(j[j[j[k]]])} \approx_w I + \frac{1}{4}I + \frac{1}{8}I + \frac{1}{8}\widehat{T}^{-k},$$

etc. Take a sequence n_i such that

$$\widehat{T}^{n_i} \to_w \frac{1}{2}I + \frac{1}{4}I + \frac{1}{8}I + \ldots = I.$$

Since

$$\widehat{T}^{n_i+m} \to_w \widehat{T}^m,$$

we obtain

$$\widehat{T}^{-h[n_i]} \approx_w \frac{1}{2}I + \frac{1}{2}\widehat{T}^{n_i} \approx_w \frac{1}{2}I + \frac{1}{2}\widehat{T}^m.$$

If the weak closure of powers of an ergodic transformation contains limits of the form $\frac{1}{2}(I + \widehat{T}^m)$, then the closure contains all admissible polynomials (see the proof in [16]). In view of the preceding, the spectra of the symmetric tensor powers $\widehat{T}^{\odot n}$ are simple. The transformation T is rigid ($\widehat{T}^{n_i} \to I$ for some sequence $n_i \to \infty$); this fundamentally distinguishes this construction from the Chacon transformation with $s_j = 1$, which has the minimal self-joining property. □

The question is, will the $(0, [rtj], 0)$-construction be simple in the sense of joining theory? If this were the case, then its centralizer would be continual, but for $s_j = j$ the centralizer is trivial. Let us prove this. If

$$\widehat{T}^{n_i} \to I, \quad h_{j(i)} \le n_i < h_{j(i)+1},$$

then, owing to the spacers, parts of the space are shifted by the spacer sizes, whence it follows that

$$\widehat{T}^{n_i - j(i)} \to I, \ \widehat{T}^{n_i - j(i) - 1} \to I.$$

But then $\widehat{T}^{(n_i - j(i)) - (n_i - j(i) - 1)}$ converges to I, and therefore, $\widehat{T} = I$.

For $s_j = j^2$, we obtain

$$\widehat{T}^{n_i - (j(i)+1)^2} \to I, \quad \widehat{T}^{n_i - (j(i)+2)^2} \to I, \quad \widehat{T}^{n_i - (j(i)+3)^2} \to I,$$
$$\widehat{T}^{-(j(i)+1)^2 + (j(i)+2)^2} \to I, \quad \widehat{T}^{-(j(i)+2)^2 + (j(i)+3)^2} \to I,$$
$$\widehat{T}^{2j(i)+3} \to I, \quad \widehat{T}^{2j(i)+5} \to I,$$
$$\widehat{T}^2 = I.$$

Example of a construction with a factor. Consider the transformation given by the sequence $\bar{s}_j = (0, 2^j, 0, 0)$. We have

$$h_{j+1} = 4h_j + 2^j, \quad \frac{2^{j+1}}{h_{j+1}} < \frac{2^j}{2h_j}.$$

Consequently,

$$\sum \frac{2^j}{h_j} < \infty.$$

This means that the transformation in question acts on a space of finite measure. It contains a factor isomorphic to the odometer with dyadic-rational spectrum (with the parameters $r_j = 2$). Indeed, starting from stage j, all parameters and the height h_j are multiples of 2^j. Hence the spectrum contains the group $\{e^{\frac{\pi i k}{2^j}}\}$.

The reader might consider various sequences s_j, say, j, j^n, 2^n, and $p(j)$, where $p(j)$ is the jth prime. What are the spectral properties of the corresponding constructions, do they have factors, and is the centralizer trivial? Some of these questions have been answered in [70].

Stochastic Chacon transformation. Let $r_j = j$. Consider all possible sequences $(s_j(1), s_j(2), \dots, s_j(j-1), s_j(j))$, where the spacer heights $s_j(i)$ take the values 0 and 1 independently with the same probability. As a result, we have an ensemble of rank one constructions. The following conjecture was put forward in the paper [16]:

For almost all stochastic constructions T, the weak closure of their powers consists of Θ and operators of the form $\widehat{T}^s P^m$, where $P = \frac{1}{2}(I + \widehat{T})$.

In view of known facts about rank one transformations, the polynomial structure of the weak closure implies the minimal self-joining property.

In connection with the cited conjecture, another problem of independent interest arises.

Let $f\colon \mathbb{Z}_r \to \{0,1\}$. Set

$$P(f, m, s) = \left| \left\{ i \in \mathbb{Z}_r \colon \sum_{w=1}^{m} f(z+w) = s \right\} \right|,$$

where z and w are added modulo r, and define

$$D(f, m) = \sum_{s=1}^{m} |P(f, m, s) - P(f, m, s-1)|.$$

Statistical lemma. *For any $\varepsilon > 0$, there exists a positive integer L such that a generic function $f\colon \mathbb{Z}_r \to \{0,1\}$ satisfies the condition*

$$D(f, m) < \varepsilon r, \quad L < m < r - L$$

for all sufficiently large r.

Using the lemma, one can prove the conjecture (the scheme of proof is presented in [69]). It is hard to doubt that the lemma is true, but it has not been proved and has the status of a conjecture. However, one can readily compensate for the lack of knowledge by modifying the construction. Roughly speaking, new construction elements ensure the properties that would follow from the lemma. The new elements are rare tall Ornstein-style spacers that do not affect those time intervals where the limits $T^s P^m$ are formed but ensure the closeness to Θ on the other time intervals. Moreover, only ordinary Ornstein spacers are used on some sequence of stages.

6. Stochastic constructions

Ornstein [59] proved the mixing property for most constructions of the statistical ensemble defined by him. Let us recall its definition. Fix some sequences $r_j \to \infty$ and $H_j \to \infty$, where H_j grows very slowly compared with r_j. Consider all possible sequences of the form $a_j(i) \in \{0, 1, \dots, H_j - 1\}$. The space

$$\{0, 1, \dots, H_1 - 1\} \times \{0, 1, \dots, H_2 - 1\} \times \dots$$

is equipped with the natural probability measure (the product of uniform distributions). The constructions are specified by the parameters

$$s_j(i) = b_j + a_j(i) - a_j(i+1);$$

in what follows, we assume that $b_j = H_j$.

The properties of the construction T depend on those of the sequence $a_j(i)$. What sufficient conditions guarantee the mixing property of T? A generic sequence $a_j(i)$ for a fixed j takes a value $s \in \{0, 1, \dots, H_j - 1\}$ with frequency close to $\frac{1}{H_j}$, and the differences $a_j(i) - a_j(i+1)$ have the triangular distribution; i.e., the fraction of i for which $a_j(i) - a_j(i+p) = s$, where $s \in \{-H_j + 1, -H_j + 2, \dots, H_j - 2, H_j - 1\}$, is close to $\frac{H_j - |s|}{H_j^2}$. This situation implies the mixing property of the construction even if the frequency distribution is asymptotically allowed to differ from the triangular distribution by an arbitrary but fixed factor.

Set

$$S_j(i,p) = s_j(i) + s_j(i+1) + \ldots + s_j(i+p-1),$$

$$Q_{j,p} = \frac{1}{r_j - p} \sum_{i=0}^{r_j-p-1} \widehat{T}^{-S_j(i,p)}.$$

Let us indicate conditions ensuring the mixing property.

Theorem 6.1. *If for any $\varepsilon > 0$ one has $Q_{j,p} \approx_s \Theta$ for all sufficiently large j and all p with $0 < p < (1-\varepsilon)r_j$, then the construction T has the mixing property.*

Outline of the proof. Let $h_j \le m < h_{j+1}$. Then

$$m = ph_j + q + \sum_{i=1}^{p} s_j(i), \; 0 \le q < h_j.$$

The specific features of rank one constructions permits one to estimate $\mu(T^m A \cap B)$ by the three-domain method. The domain D_0 is the union of columns (of stage j) with numbers from 1 to p, the domain D_1 is a union of upper parts of the remaining columns (the nth floors for $q < n < h_j$), and the domain D_2 is the remaining set corresponding to the union of lower parts of the columns (the nth floors for $0 < n < q$).

Let A and B consist of floors of the tower ξ_j. Then

$$\mu(T^m A \cap B) \approx \frac{1}{r_{j-1}} \sum_{i=1}^{r_{j+1}-1} \mu\big(T^{m_j - h_{j+1}} T^{-s_{j+1}(i)} A \cap B \cap D_1\big)$$
$$+ \frac{1}{r_j - q} \sum_{i=1}^{r_j - q} \mu\big(T^k T^{-S_j(i,q)} A \cap B \cap D_2\big)$$
$$+ \frac{1}{r_j - q - 1} \sum_{i=1}^{r_j-q-1} \mu\big(T^{-h_j+k} T^{-S_j(i,q+1)} A \cap B \cap D_3\big).$$

By f and g we denote the indicator functions of the sets A and B, respectively. Let us rewrite the preceding expression as

$$(\widehat{T}^m f, g) \approx \int_D \widehat{T}^k Q_{j,1} f\, g\, d\mu + \int_{D_1} \widehat{T}^{k_1} Q_{j,p} f\, g\, d\mu + \int_{D_2} \widehat{T}^{k_2} Q_{j,p+1} f\, g\, d\mu.$$

Since $Q_{j,p} \approx_s \Theta$ by the assumption of the theorem, we obtain

$$(\widehat{T}^m f, g) \approx (\mu(D) + \mu(D_1) + \mu(D_2))(\Theta f, g) = \mu(A)\mu(B).$$

It follows that the construction T is mixing. The proof of the theorem is complete. □

Let us verify that the assumption of the theorem is satisfied for the Ornstein transformations. We have

$$S_j(i,p) = pH_j + a_j(i) - a_j(i+p),$$

and hence

$$\widehat{T}^{-pH_j} Q_{j,p} = \frac{1}{r_j - p} \sum_{i=0}^{r_j-p-1} \widehat{T}^{a_j(i) - a_j(i+p)}.$$

But

$$\big\|\widehat{T}^{pH_j} Q_{j,p} - P_j^* P_j\big\| \approx 0, \quad \text{where} \quad P_j = \frac{1}{H_j} \sum_{i=0}^{H_j-1} \widehat{T}^i.$$

It is well known that

$$P_j \approx_s \Theta, \quad P_j^* P_j \approx_s \Theta$$

for ergodic T. We conclude that

$$Q_{j,p} \approx_s \widehat{T}^{pH_j}\Theta = \Theta.$$

Let us explain in what sense the distribution of $S_j(i,p)$ can differ from the uniform or triangular distribution but still ensure the mixing property of the construction.

Theorem 6.2. *Let R_j be a sequence of Markov operators commuting with a weakly mixing construction T, and let $R_j \to_s \Theta$. If for some $a > 0$ one has $R_j \geq aQ_{j,p}$ and $R_j \to_s \Theta$, for $0 < p < (1-\varepsilon)r_j$ for all sufficiently large j, then the construction T is mixing.*

Proof. One has

$$R_j^* R_j \to_s \Theta, \quad a^2 Q_{j,p}^* Q_{j,p} + \ldots \to_s \Theta.$$

Since the operator Θ is indecomposable in the Markov centralizer of the operator $\widehat{T}$ (because $\mu \times \mu$ is ergodic) and the $Q_{j,p}$ lie in the centralizer, we obtain

$$Q_{j,p}^* Q_{j,p} \to_w \Theta, \quad Q_{j,p} \to_s \Theta.$$

An application of the preceding theorem completes the proof. □

7. Genericity of rank one, nonstandard mixing, and nongeneric transformations

The genericity of rank one in the space Mix was proved in [2] based on Bashtanov's result on the proximity of any conjugacy class to a Bernoulli transformation and S. V. Tikhonov's theorem stating that Bernoulli transformations are dense in Mix. The genericity of rank one means the following: the set of mixing transformations is a dense G_δ set in Mix.

Our approach is different: we directly show that the conjugacy class of the Ornstein construction is dense in the space Mix. Recall that transformations S and T are close in Mix if the powers $\widehat{S}^k$ and $\widehat{T}^k$ are close in the weak operator topology for all k. Note that the same definition applies for the case of spaces with infinite measure.

Let S be a mixing transformation, and let T' be a given Ornstein construction. We need to find a conjugation R such that not only $T = R^{-1}T'R$ is close to S in the weak topology, but the same is also true for all of their powers. This means that once the powers of S^i become close to Θ, we can forget about them and switch our attention to ensuring that the powers of T^i be close to Θ as well. This will imply that the transformation S and the conjugate of T are close in Mix.

Take the Rokhlin–Halmos tower of height $N+a$ of the transformation S with a small remainder and a tower ξ' of height $h = h_j$ of the rank one transformation T' for a very distant stage j. Assume that $h = Nq$, $q \gg N \gg a \gg 1$. Mark the floors with numbers $iN + a(i)$, $i = 0, 1, 2, \ldots, q-1$, in the tower ξ'. Now wind the tower ξ' around the tower ξ of the transformation S in such a way that the marked floors be on the first floor of ξ and the pieces of orbits of length $N + a(i+1) - a(i)$ between the marked floors be aligned with pieces of orbits of S of the same length.

The winding is the desired conjugation, and we obtain a conjugate transformation T that coincides with S on the first $N-a$ floors. One can readily ensure that the difference of the powers T^i and S^i be small for $0 < i < M$ and $S^i \approx \Theta$ for $i > M$. It remains to verify that $T^i \approx \Theta$ for all $i > M$. Let us see, say, what happens for $i \approx N/2$. On the upper half D of the tower ξ, the power T^i coincides with S^i (neglecting a small set of measure comparable with a/N). But $S^i \approx_w \Theta$, and so T^i is close to Θ on D. This should be understood as follows: for sets A and B given in advance,

$$\mu(T^i A \cap B \cap D) \approx \mu(A)\mu(B)\mu(D).$$

The mixing on the lower half is ensured by the virtual spacers $a + a(i+1) - a(i)$. Here one should have in mind that the a set given in advance is uniformly distributed in the tower ξ and that q in the relation $h = Nq$ is taken to be fairly large, $q \gg 2^N$. For $i > N$ we forget about S^i completely, and the powers of T^i mix a given set of sets chosen in advance owing to the virtual and then real spacers. This is a brief outline of how to prove that the rank one Ornstein transformation is dense in Mix. From this, one can also obtain the assertion that rank one is generic in the space Mix.

In a similar way, one proves that the Ornstein flow is dense in the space of mixing flows.

Theorem 7.1. *Rank one flows are generic in the space of mixing flows. Hence the minimal self-joining property is generic as well.*

In the case of spaces with infinite measure, the proof of the genericity of rank one can be simplified by invoking the idea of Sidon spacers.

On the isomorphism problem for mixing constructions. In [68], a sufficient condition is given for two mixing constructions T and T' not to be isomorphic in the case of a probability space.

Theorem 7.2. *If the tower heights h_j and h'_j for respective rank one mixing constructions T and T' are comparable but their ratio does not tend to* 1*, then T and T' are not isomorphic.*

A similar statement is true for rank 1 mixing flows. Let us state a corollary in [11].

Theorem 7.3. *A rank one mixing flow T_t is not isomorphic to the flow T_{at} for $a > 1$.*

Nonstandard mixing. If the tower height ratio tends to 1, it would be of interest to find some properties or invariants distinguishing one mixing construction from the other.

Let T be a rank one mixing transformation with the corresponding sequence of partitions ξ_j. Consider sets A_j and B_j consisting of atoms of the partition ξ_j and asymptotically uniformly distributed in the tower. This means that most intervals consisting of $L = L(\varepsilon)$ consecutive floors contain approximately $\mu(A_j)L$ (with accuracy εL) floors lying in A_j. We say that such sequences A_j are *proper*. (When working with such sequences, one can impose the condition $\mu(A_j) \to a > 0$.)

The transformation is *well mixing* if

$$\sup_{m > h_j} \mu(T^m A_j \cap B_j) - \mu(A_j)\mu(B_j) \to 0, \quad j \to \infty,$$

for any proper A_j and B_j. This definition is inspired by nonstandard analysis: here the transformation mixes not only standard sets ($A_j = A$ and $B_j = B$), but also some nonstandard sets.

Almost all Ornstein transformations are well mixing, but the staircase constructions are not. The latter implement the case in which $\mu(A_j) \to \frac{1}{2}$ and

$$\mu(T^{2h_j} A_j \cap A_j) - \mu(A_j)^2 \to \frac{1}{4}.$$

For such A_j one can take the union of odd floors in the tower of the partition ξ_j.

Nongeneric properties. The next question makes sense for both the space Aut and the space Mix: *Let K be a compact set of transformations. Is it true that there exists a dense G_δ set Y that does not meet K^{Aut}?*

By K^{Aut} we denote the union of all conjugacy classes in the group Aut that have a representative in $K \subset \mathrm{Aut}$. We fix a metric dist defining the weak operator topology.

Theorem 7.4. *Let $K \subset \mathrm{Aut}$ be a compact set. Assume that for some $r > 0$ and for any positive integer j there exists an $m > j$ such that $\mathrm{dist}(\widehat{T}^m, \Theta) > r$ for all transformations $T \in K$. Then there exists an everywhere dense G_δ-set Y disjoint from K^{Aut}.*

Proof. Let $m(T, j)$ be the minimum $m > j$ such that $\mathrm{dist}(\widehat{T}^m, \Theta) > r$. Since K is compact, it follows that $m(T, j)$ is a bounded function on K. Let $M(j)$ be the maximum of $m(T, j)$. Consider the sets $F_j = \{j, j+1, \dots, M(j)\}$. It was proved in [15] that there exists an everywhere dense G_δ-set Y such that for each $S \in Y$ there exists a mixing subsequence of sets $F_{j(k)}$, $j(k) \to \infty$; i.e.,

$$\mathrm{dist}(\widehat{S}^{m(k)}, \Theta) \to 0, \quad k \to \infty,$$

for $m(k) \in F_{j(k)}$. Such a transformation S and any conjugate of it do not belong to K, because otherwise

$$\exists m \in F_{j(k)} \colon \ \mathrm{dist}(\widehat{S}^m, \Theta) > r$$

for all k. The proof of the theorem is complete. □

One can give various examples of compact sets satisfying the assumptions of the theorem. The simplest case is given by a finite set of nonmixing transformations. Well-known examples are provided by the permutations of k segments. As a consequence, we obtain the result in [36] that permutations of a finitely many segments are not generic transformations. Our proof is shorter, because we have not cared about estimating the numbers $M(j)$. The condition $\mathrm{dist}(\widehat{T}^m, \Theta) > r$ for permutations is satisfied owing to the well-known partial rigidity property.

It is an open question whether a permutation of finitely many rectangles can have the mixing property.

8. Algebraic parameters instead of random ones

Let $r_j \to \infty$ be a sequence of primes. Take a generator q_j of the multiplicative group of the field $\mathbf{F}_{r_j}$, which is identified with the set of positive integers $0, 1, \dots, r_j - 1$. Set

$$s_j(i) = r_j + \{q_j^i\} - \{q_j^{i+1}\},$$

where $\{q\}$ is the positive integer (remainder) corresponding to an element q of our field. Now consider a rank one construction T with the parameters defined above. We need to establish the mixing property. Set

$$S_j(i,p) = s_j(i) + s_j(i+1) + \ldots + s_j(i+p-1),$$

$$Q_{j,p} = \frac{1}{r_j - p} \sum_{i=0}^{r_j - p - 1} \widehat{T}^{-S_j(i,p)}.$$

We have

$$S_j(i,p) = pr_j + \{q_j^i\} - \{q_j^{i+p}\},$$

$$\widehat{T}^{-pr_j} Q_{j,p} = \frac{1}{r_j - p} \sum_{i=0}^{r_j - p - 1} \widehat{T}^{\{q_j^i\} - \{q_j^{i+p}\}}.$$

However, the function $\{q_j^i\}-\{q_j^{i+p}\}$ is injective as a function of i,

$$\{q_j^i\}-\{q_j^{i+p}\}=\{q_j^k\}-\{q_j^{k+p}\},\quad q_j^i-q_j^{i+p}=q_j^k-q_j^{k+p},\quad q_j^i=q_j^k,\quad i=k.$$

Set

$$R_j=\frac{1}{r_j-p}\sum_{i=pr_j}^{(p+1)r_j-p}\widehat{T}^i.$$

Since $R_j\approx_s\Theta$, to establish the mixing property it remains to verify that our construction has the weak mixing property and apply Theorem 6.2.

Let us give an example due to M. S. Lobanov and the author. Consider the trace mapping

$$\operatorname{tr}\colon \mathbf{F}_{b^n}\to\mathbf{F}_b,\quad \operatorname{tr}(q)=\sum_{s=0}^{n-1}q^{b_j^s}.$$

Set

$$r_j=b_j^{n_j}-1\to\infty,\quad a_j(i)=\operatorname{tr}(q_j^i)=\sum_{s=0}^{n-1}q_j^{b^s},$$
$$s_j(i)=b_j+\{a_j(i)\}-\{a_j(i+1)\}.$$

The parameters r_j and $s_j(i)$ determine a construction T, for which the mixing property is established by analogy with the mixing property of Ornstein transformations. Set

$$P_j=\frac{1}{b_j}\sum_{i=0}^{b_j-1}\widehat{T}^i,\quad R_j=\widehat{T}^{-pb_j}P_j^*P_j$$

and apply Theorem 6.2. In this case, the weights in the sums $Q_{j,p}$ obey the triangular distribution, and we obtain a new class of mixing constructions. It may be of interest to study the spectra of such constructions.

Simple spectrum of tensor products. Parameters specified via the trace function $\operatorname{tr}\colon\mathbf{F}_{2^n}\to\mathbf{F}_2$ were used in [62] to construct a flow T_t such that the products $T_1\times T_t$ have simple spectrum for any $t>1$.

A similar construction satisfies the stronger condition that the products

$$T_{t_1}\times T_{t_2}\times T_{t_3}\times\dots$$

have simple spectrum for any pairwise distinct $t_i>0$. This is because that such products have all kinds of weak limits of the form

$$P_b\otimes P_b\otimes P_b\otimes\dots,\qquad 4P_b=\widehat{T}_{-b}+2I+\widehat{T}_b,\quad b\in\mathbb{R}.$$

The existence of these limits ensures the simplicity of the spectrum of the products provided that the original flow T_t has simple spectrum. The flow T_t uses spacers of the form

$$s_j(i)=a_j(i)-a_j(i+1)+b_j,\quad a_j(i)=|\operatorname{tr}(q_j^i)|b_j,$$

where the sequence b_j takes each rational value infinitely many times. The methods in [16] permit constructing a mixing flow with a similar spectral property.

9. Staircase constructions

A staircase construction is defined by a sequence of spacers

$$s(i) = i,\ i = 1, 2, \dots, r_j, \quad r_j \to \infty.$$

Adams [26] found an original method for proving the mixing property of such transformations for the case in which $r_j^2/h_j \to 0$.

For a staircase construction, one can readily establish mixing for a sequence $\{m_j\}$ provided that $m_j \in [h_j, Ch_j]$ for a given number $C > 1$. For example, using the ergodicity of the power T^p, we obtain

$$\mu(T^{ph_j}A \cap B) \approx \frac{1}{r_j - p} \sum_{i=0}^{r_j-p-1} \mu(T^{-pi-k(p,j)}A \cap B) \approx \mu(A)\mu(B).$$

In the general case, the situation is similar to that considered in Sec. 5, but here one has to deal with averagings of the form

$$\frac{1}{N_j} \sum_{i=0}^{N_j-1} \mu(T^{-d_j i + k_j} A \cap B \cap D),$$

where the d_j are, generally speaking, unbounded.

Thus, we need to show that

$$P_j = \frac{1}{N_j} \sum_{i=0}^{N_j-1} \widehat{T}^{-d_j i} \approx_w \Theta.$$

Fix a large L. If $d_j \in [h_{p(j)}, 2h_{p(j)}]$, then

$$id_j \in [h_{p(j)}, 2Lh_{p(j)}], \quad 0 < i < L. \tag{9.1}$$

Hence it follows that

$$\widehat{T}^{d_j}, \widehat{T}^{2d_j}, \dots, \widehat{T}^{(L-1)d_j} \approx_w \Theta. \tag{9.2}$$

Set

$$Q_L = \frac{1}{L} \sum_{i=0}^{L-1} \widehat{T}^{d_j i}.$$

By (9.2), $Q_L^* Q_L \approx_w \Theta$, which is equivalent to $Q_L \approx_s \Theta$. Now we obtain

$$P_j \approx_s P_j Q_L \approx_s P_j \Theta \approx_s \Theta.$$

However, (9.1) may not hold. Then Adams finds a positive integer a such that

$$aLd_j \ll N_j, \qquad ad_j \in [h_{p(j)}, 2h_{p(j)}], \quad \frac{1}{L} \sum_{i=0}^{L-1} \widehat{T}^{iad_j} =: A_L \approx_s \Theta,$$

which leads to the desired result,

$$P_j \approx_s P_j A_L \approx_s \Theta.$$

A specific feature of the Adams method is that mixing at stage j uses information obtained at the previous stages. The proof uses the restriction $r_j^2/h_j \to 0$ on the growth of the sequence r_j. This condition permits one to control mixing using only three averagings corresponding to three regions of the phase space. We described them in Sec. 5. If $r_j^2/h_j \to \infty$, then infinitely many domains $D, D_1, \dots, D_k, \dots$ *arise*, to each of which there corresponds an averaging operator of the form

$$Q_{L,k} = \frac{1}{L} \sum_{i=0}^{L-1} \widehat{T}^{i(d_j+k)}.$$

Since T is a weakly mixing transformation, it follows that

$$\widehat{T}^{d_j+k}\widehat{T}^{2(d_j+k)}\dots\widehat{T}^{(L-1)(d_j+k)} \approx_w \Theta$$

for most numbers k. Therefore, $Q_{L,k} \approx_s \Theta$ for most numbers k. This shows that the mixing effect is observed on most domains, and therefore, the construction has the mixing property. This approach is presented in detail in [67]. The authors of [37] also worked on the mixing problem for staircase constructions, and they obtained a number of generalizations of Adams' result.

The sequence r_j can grow so fast that the phase space of the staircase construction will have infinite measure. The next statement deals with spaces with finite as well as infinite measure.

Theorem 9.1. *If* $\frac{r_j}{h_j} \to 0$*, then the staircase construction has the mixing property.*

Note that the case of $r_j \sim h_j$ is a difficult problem, where the convergence $\widehat{T}^i \to_w 0$ has not been proved and the above methods do not work.

10. Explicit mixing construction with double spectrum

The interest in staircase constructions and their modifications was due to the problem on the homogeneous spectrum of a transformation in the class Mix. The paper [64] indicated a class of transformations and proved that there exist desired constructions T in this class such that $T \times T$ has a double spectrum. What exactly these conditions are remained unclear.

Set $\bar{s}_j = (1, 2, \dots, r_j - 2, r_j - 1, 0)$ for all stages starting from some stage. We assume that, starting from some moment, the sequence r_j takes a value r successively 2^{4r} times in a row, then $2^{4(r+1)}$ times in a row r_j takes the value $r + 1$, etc. We say that such a growth of the sequence r_j is slow. Our aim is to prove the following statement.

Theorem 10.1. *If the sequence* r_j *of the construction* T *has slow growth, then* $\widehat{T} \otimes \widehat{T}$ *has homogeneous double spectrum.*

Fix the indicator function f of some floor. It is well known that f is a cyclic vector. Let us prove that for our construction T the vectors $\widehat{T}^s f \otimes f + f \otimes \widehat{T}^s f$ belong to the cyclic space $C_{f\otimes f}$ of the operator $\mathbf{T} = \widehat{T} \otimes \widehat{T}$ for all $s > 0$. This means that the symmetric power $\widehat{T} \odot \widehat{T}$ has simple spectrum, which implies the multiplicity 2 for $\widehat{T} \otimes \widehat{T}$.

Assume that the following inequalities hold for some sequence $M(r) \to \infty$, $r \to \infty$, for all m and n with $1 \le m < n \le M(r)$:

$$\left|(\mathbf{T}^{p(m,r)}F, \mathbf{Q}_r F) - (\mathbf{Q}_r F, \mathbf{Q}_r F)\right| < \varepsilon_r,$$

$$\left|(\mathbf{T}^{p(m,r)}F, \mathbf{T}^{p(n,r)} F) - (\mathbf{Q}_r F, \mathbf{Q}_r F)\right| < \varepsilon_r,$$

where

$$\mathbf{Q}_r = \frac{1}{r}\sum_{i=0}^{r-1} \mathbf{T}^i.$$

We set

$$P_r = \frac{1}{M(r)}\sum_{m=1}^{M(r)} \mathbf{T}^{p(m,r)}$$

and obtain

$$\|\mathbf{P}_r F - \mathbf{Q}_r F\|^2 \ \le\ \frac{\|F\|^2}{M(r)} + \varepsilon_r.$$

Let $M(r)$ and $p(m,r)$ be chosen in such a way that the right-hand side of the last inequality does not exceed a value comparable with 2^{-r}. (We explain later how to choose them.) In this case, we say that the vectors $\mathbf{P}_{M(r)}F$ and $\mathbf{Q}_rF$ are very close and write $\mathbf{P}_{M(r)}F \approx_r \mathbf{Q}_rF$.

From the preceding and the relation

$$I \otimes \widehat{T}^r + \widehat{T}^r \otimes I = (r+1)\mathbf{Q}_{r+1} - r\mathbf{Q}_r - r\mathbf{T}\mathbf{Q}_r + (r-1)\mathbf{T}\mathbf{Q}_{r-1},$$

we obtain

$$(10.1) \quad (I \otimes \widehat{T}^r + \widehat{T}^r \otimes I)F \approx_r \big((r+1)\mathbf{P}_{M(r+1)} - r\mathbf{P}_{M(r)} - r\mathbf{T}\mathbf{P}_{M(r)}\mathbf{Q}_r + (r-1)\mathbf{T}\mathbf{P}_{r-1}\big)F := \mathbf{V}_rF.$$

Taking the vector $f \otimes f$ for F, we obtain

$$\widehat{T}^r f \otimes f + f \otimes \widehat{T}^r f \approx_r \mathbf{V}_r(f \otimes f).$$

Note that the norm of the operator $\mathbf{V}_r$ defined above is $4r$, and so it takes very close vectors to very close ones. Substituting the vector $\widehat{T}^r f \otimes f + f \otimes \widehat{T}^r f$ for F and $r+s$ for r into (10), we obtain

$$(\widehat{T}^{r+s} \otimes I + I \otimes \widehat{T}^{r+s})(\widehat{T}^r f \otimes f + f \otimes \widehat{T}^r f) \approx_{r+s} \mathbf{V}_{r+s}(\widehat{T}^r f \otimes f + f \otimes \widehat{T}^r f) \approx_r \mathbf{V}_{r+s}\mathbf{V}_r(f \otimes f).$$

We have

$$\mathbf{T}^r(\widehat{T}^s f \otimes f + f \otimes \widehat{T}^s f) \approx_r \mathbf{V}_{r+s}\mathbf{V}_r(f \otimes f) - (\widehat{T}^{2r+s} f \otimes f + f \otimes \widehat{T}^{2r+s} f).$$

The right-hand side is the difference of two vectors, the first of which belongs to the cyclic space $C_{f\otimes f}$ and the second is at a small distance comparable with 2^{-2r-s} from $C_{f\otimes f}$. It turns out that for any $s > 0$ the vector $\widehat{T}^s f \otimes f + f \otimes \widehat{T}^s f$ lies at a distance at most of the order of 2^{-r} from $C_{f\otimes f}$; therefore, the distance is zero.

It remains to explain how to choose $M(r)$ and $p(m,r)$, $1 \le m \le M(r)$.

Let the monotone sequence r_j take the value r successively 2^{3r} times in a row starting from the number j_r. Set $M(r) = 2^{2r}$ and $p(m,r) = h_{j_r+2^{2r}m}$, where $m = 1, 2, \dots, 2^r$. A straightforward verification shows that ε_r tends to zero much faster than 2^{-r}. Thus, for this construction we have

$$\|\mathbf{P}_rF - \mathbf{Q}_rF\| \ \le 2^{-r}.$$

It is true for some staircase constructions that their symmetric tensor powers have simple spectrum [16]. The conjecture that staircase constructions with logarithmic growth of the sequence r_j possess the same property seems to be very plausible.

The reader may notice that we have chosen logarithmic growth to avoid computations. In fact, it can be replaced with a power law. For example, for $r_j^4 \sim j$ a similar reasoning leads to the conclusion that the vectors $\widehat{T}^s f \otimes f + f \otimes \widehat{T}^s f$ belong to the cyclic space $C_{f\otimes f}$.

It is of interest to know what the multiplicity of the spectrum of $\widehat{T} \otimes \widehat{T}$ (and other tensor powers) is for the staircase construction T if $r_j = [rtj]$, $r_j = j$, and $r_j = j^2$.

Remark. There exist known rank one transformations R for which $R \times R$ has the maximum spectral multiplicity 2^n. They can be constructed as follows. Find n transformations for which the multiplicity of the spectrum of the Cartesian square is 2, their Cartesian product R has rank one, and the maximum multiplicity of the spectrum of $R \times R$ is 2^n.

11. Infinite transformations and self-similar constructions

Following [70], consider the simplest self-similar construction T given by the parameters $\bar{s}_j = h_j(0,1)$. Since

$$\widehat{T}^{h_j} \to_w \frac{1}{2} I,$$

it follows that the spectrum of this construction is singular, and the convolution powers of the spectral measure are mutually singular. Note that

$$\widehat{T}^3 \cong \widehat{T} \oplus \widehat{T} \oplus \widehat{T}.$$

The power T^3 is not ergodic, and the space X splits into three invariant sets on each of which T^3 is similar to T. It is in this sense that we say that the transformation is self-similar. That is why the spectrum $\widehat{T} \otimes \widehat{T}^3$ is singular. Will the spectrum of the product $\widehat{T} \otimes \widehat{T}^2$ be singular? We leave this question for further study.

Consider another construction with parameters $\bar{s}_j = h_j(0,3)$. The product $T \otimes T^2$ is dissipative and hence has countably multiple Lebesgue spectrum. The product $T \otimes T^5$ is conservative and has singular spectrum. For similar, but not self-similar examples, see [22]. The simplest among them have spacers of the form $\bar{s}_j = (0, s_j)$, $s_j \gg h_j$.

The paper [5] considers the construction with parameters

$$\bar{s}_j = h_j(1,5), \quad h_j = 8^j.$$

and describes the semigroup of weak power limits for it. The semigroup consists of the zero operator and operators of the form $2^{-m}\widehat{T}^s$, $m = 0, 1, 2, \ldots$ By the results in [22], this construction has a trivial centralizer.

Asymmetry of the past and future. One can readily reveal the asymmetry of infinite self-similar constructions. In the case of a finite measure, this requires considerably more ingenuity; e.g., see [9].

Theorem 11.1. *The transformation with parameters $\bar{s}_j = h_j(0,1,2)$ is not isomorphic to the inverse of itself.*

Proof. One can readily verify that if A is a ξ_i-measurable set of finite measure, then

$$\mu(A \cap T^{h_j} A \cap T^{3h_j} A) = \frac{\mu(A)}{3}, \quad \mu(A \cap T^{2h_j} A \cap T^{3h_j} A) = 0$$

for all $j > i$, whence it follows that

$$\mu(A \cap T^{-h_j} A \cap T^{-3h_j} A) = 0.$$

If T and S are conjugate, then

$$\mu(A \cap S^{h_j} A \cap S^{3h_j} A) \to \frac{\mu(A)}{3},$$

and if T^{-1} and S are conjugate, then $\mu(A \cap S^{h_j} A \cap S^{3h_j} A) \to 0$. Therefore, T and T^{-1} cannot have a common conjugation S, and so they are not isomorphic. □

Self-similar flows. A rank one flow is defined in a similar way. The stage j tower is identified with a rectangle of height h_j vertically cut into $r_j > 1$ columns of the same width; rectangles of heights $s_j(i) \in \mathbb{R}$ are built above them. The point moves vertically at a constant speed; once it reaches the top of the ith spacer column, it is at the base of the $(i+1)$st column. We mentally add the spacer columns into one column, which is now the stage $j+1$ rectangle, etc. (see [62] for a more detailed description). The flow T_t with spacers $\bar{s}_j = h_j(0, \alpha)$ is similar to the flow $T_{(2+\alpha)t}$ in the sense that they are conjugate in the group of nonsingular transformations. Gaussian spacers over such a flow are an addition to the collection of self-similar flows.

Note that infinite rank one transformations of Chacon type have found an interesting application to the isomorphism problem for Poisson suspensions with coinciding spectra [46]. The Poisson suspension changes when the density of the measure is replaced with a nonisomorphic one but having the same spectrum. In this vein, it would be of interest to study spacers over self-similar flows proposed above. The aim is to obtain a Poisson flow P_t that is not isomorphic to $P_{\alpha t}$ even though they have the same singular spectrum.

Using modifications of the constructions in [22], we can construct a Poisson flow satisfying the assumptions of the following theorem.

Theorem 11.2. *Given disjoint countable sets C and C', there exists a weakly mixing flow T_t such that the spectrum of the products $\widehat{T}_1 \otimes \widehat{T}_c$ is singular for $c \in C$, $c > 1$, but has a Lebesgue component for $c \in C'$, $c > 1$.*

The spectral measure σ of such a flow has the following property: the product $\sigma \times \sigma$ has both singular and nonsingular projections onto the diagonal in $\mathbb{R}^2$ depending on the projection angle. The following terminology suggests itself: $\sigma \times \sigma$ is transparent along some directions and gives full shadow along other directions.

Mixing constructions. If the parameters are chosen to satisfy

$$h_j \ll s_j(1) \ll s_j(2) \ll \ldots \ll s_j(r_j - 1) \ll s_j(r_j)$$

then the corresponding construction has the mixing property. Indeed, for Sidon constructions it follows from the condition $\sum_j \frac{1}{r_j} < \infty$ that the points in $E_1 \times E_1$ eventually stop returning to $E_1 \times E_1$, and we obtain $X \times X = \bigsqcup_{i \in \mathbb{Z}} (T \times T)^i Y$ for a measurable set Y of infinite measure. It is of interest to find out what the spectral type of the measure σ can be for such transformations T. Finding an infinite rank one transformation with Lebesgue spectrum would solve the famous Banach problem, mentioned by Ulam [71], on a transformation with simple Lebesgue spectrum.

Thouvenot's problem. We see that the Cartesian squares of Sidon constructions are dissipative transformations with an infinite wandering set, and hence they are isomorphic. The following question arises in connection with Thouvenot's problem (see [20]): what invariants can distinguish Sidon constructions? Let us give an example.

Let $F_j \subset \mathbb{N}$ be a sequence of finite sets. We say that a transformation T of an infinite measure space belongs to the class $\alpha \in [0, 1]$ if

$$\limsup_j \mu\Big(\cup_{n \in F_j} T^n A \mid A \Big) = \alpha$$

for any set A of finite measure. Obviously, α is an invariant. One can find a sequence of integer intervals F_j such that for each $\alpha \in [0, 1]$ there exists a construction of the class α with dissipative Cartesian square. In the case of rank one transformations on a probability space, it is still not known whether the isomorphism of Cartesian powers of transformations implies an isomorphism of the transformations themselves.

12. Concluding remarks and questions

There are a large number of open questions associated with rank one transformations; some of them are stated below.

1. General questions about rank one transformations.

1.1. Is it true that any ergodic self-joining ν of a rank one transformation T is the limit of off-diagonal measures, $\Delta^{k(j)} \to \nu$? In other words, is it true that an indecomposable Markov operator P commuting with $\widehat{T}$ lies in the weak closure of powers of $\widehat{T}$? *J. King*

We know that for an ergodic self-joining ν there exists a sequence $k(j)$ such that $\Delta^{k(j)} \to \eta \geq \frac{1}{2}\nu$. Conjecture: the coefficient $\frac{1}{2}$ is sharp.

1.2. Does a nonrigid rank one transformation without factors have the MSJ property?

J.-P. Thouvenot

1.3. Is it true that the centralizer of an infinite rank one transformation lies in the weak closure of powers of the transformation? *E. Roy*

2. Weak multiple mixing WM(n)**.** In joining theory, the following question related to the multiple mixing problem has long been open: *Is it true that a weakly mixing transformation with zero entropy does not admit nontrivial pairwise independent self-joinings?*

Although there has been significant progress (see [43]), one question remains open for nonmixing rank one actions. The answer is affirmative for a rank one transformation with the weak multiple mixing property of order 3, WMix(3). Recall that this property means the following: if k_i, m_i, and n_i are sequences such that the convergence

$$\mu(A \cap T^{k_i}B \cap T^{m_i}C \cap T^{n_i}D) \to \mu(A)\mu(B)\mu(C)\mu(D)$$

holds for all tuples of sets $A, B, C, D \in \mathcal{B}$ of which at least one coincides with X, then the convergence holds for arbitrary tuples $A, B, C, D \in \mathcal{B}$. The property WM($n$) is defined in a similar way.

Theorem 12.1. *For rank one transformations with any fixed* $k > 0$*, the property* WMix($2k + 1$) *implies the weak multiple mixing property* WMix(m) *for all* $m > 1$ *and the absence of nontrivial pairwise independent self-joinings.*

This fact follows from the results in [10, 11].

3. Homoclinic rank one transformation groups. The homoclinic group $H(T)$ of an automorphism T was introduced by M. I. Gordin as follows:

$$H(cc\widehat{T}) = \bigl\{\widehat{S} \in \mathrm{Aut}(\mu)\colon \widehat{T}^{-n}\widehat{S}\,\widehat{T}^{n} \to I,\ n \to \infty\bigr\},$$

here strong operator convergence to I is meant. Some generalizations of this concept turned out to be meaningful. In particular, they are of interest in connection with the general question: to what extent do rank one actions differ from Gaussian and Poisson actions (see [7]). The point is that the latter have extensive homoclinic groups [21], and rank one actions in particular cases have a trivial homoclinic group. Let us give the definitions.

The weakly homoclinic group is defined as follows:

$$WH(\widehat{T}) = \left\{\widehat{S} \in \mathrm{Aut}(\mu)\colon \frac{1}{N}\sum_{i=1}^{N}\widehat{T}^{-i}\widehat{S}\,\widehat{T}^{i} \to I,\ N \to \infty\right\}.$$

The group $wH(\{T_g\})$ of the action $\{T_g\}$ (it may also be called weakly homoclinic) consists of transformations S such that $\widehat{T}_{g_i}^{-1}\widehat{S}\widehat{T}_{g_i} \to I$ for all sequences $\widehat{T}_{g_i} \to_w \Theta$.

Associated with an infinite set $P \subseteq \mathbb{Z}$ and the transformation T is the group

$$H_P(T) = \bigl\{S \in \mathrm{Aut}(\mu)\colon \widehat{T}^{-n}\widehat{S}\,\widehat{T}^{n} \to I,\ n \in P,\ n \to \infty\bigr\}.$$

Note the inclusions

$$H(T) \subseteq wH(\{T^n\}) \subseteq WH(T), \quad H(T) \subseteq H_P(T).$$

For mixing rank one transformations T, the group $WH(T)$ is trivial [21], but nothing is known about the groups $H_P(T)$ for sparse sets P. If the group WH of a rank one transformation is ergodic, then the transformation is rigid [21].

What are the groups WH and H_P for generic rigid weakly mixing rank one transformations?

Can a rank one transformation have an ergodic weakly homoclinic group WH?

The negative answer would confirm the conjecture that Poisson suspensions do not have rank one.

4. Constructions with a large scatter of parameters. The methods discussed in this article do not apply to a vast, one might say, overwhelming variety of rank one constructions. By analogy with the Pascal transformation introduced by Vershik [3], one can propose to study the spectral and metric properties of a binomial rank one transformation given by the parameters $r_j = j+1$ and $s_j(i) = C_j^i$. When thinking of this example, one finds an abundance of constructions for which the question as to whether the spectrum is continuous, which is so easy for bounded constructions, is a serious problem. It is probably useful to consider products of the form

$$Q_J = \prod_{j\in J} P_j, \quad P_j = \frac{1}{j+1}\sum_{i=0}^{j} \widehat{T}^{s_j(i)}$$

for large finite sets J. (Here an association with Riesz products arises.) If large sums of random binomial coefficients are well distributed, the situation arises

$$\widehat{T}^{-\sum_{j\in J} h_j} \approx_w \widehat{T}^n Q_J \approx_w \Theta,$$

which will give the desired continuity of the spectrum. The last approximation is difficult. The first is obvious under reasonable restrictions on J.

5. Several problems about rank one constructions.

5.1. For an infinite staircase construction, under the condition $\frac{r_j}{h_j} \to 0$ and $r_j \to \infty$, the mixing property [67] holds. Does the staircase construction for r_j have the mixing property for $r_j = h_j$? This raises nontrivial questions about the intersections of quadratic subsets of $\mathbb{N}$ with their translates.

5.2. Do the symmetric powers of bounded weakly mixing constructions have simple spectrum?

5.3. Is it true that constructions with spacers $(0, s_j, 0)$ are simple for $s_j \sim j^\alpha$, $\alpha > 0$, that is, $\mu\times\mu$ and measures of the form Δ_S are the only ergodic self-joinings of order 2?

For $s_j = j$, the weak closure contains many polynomial limits, which speaks in favor of the minimality of the spectrum.

For $s_j = j^n, n > 1$, only limits of the form $aI + (1-a)\Theta$ are obvious, and the question concerning the existence of limits of the form $aI + b\widehat{T}^k + \dots$ may turn out to be a difficult problem.

5.4. Let $p(j)$ be the jth prime. Deep facts of number theory imply that the convolution powers of the spectral measure of the construction T with spacers $(0, p(j), 0)$ are disjoint. Nontrivial weak limits are indicated in [70] as a consequence of new facts about primes [72]. They ensure the weak mixing property. It follows from the classical results due to I. M. Vinogradov that the strong convergence

$$\frac{1}{N}\sum_{j=1}^{N} \widehat{T}^{p(j)} \to \Theta$$

holds for a weakly mixing transformation T. Since Θ is indecomposable in the Markov centralizer of the operator $\widehat{T}$, which is equivalent to the weak mixing property, one has $T^p \approx_w \Theta$ for most primes p, whence we obtain

$$\widehat{T}^{-h_j} \approx_w \frac{1}{2}I + \frac{1}{3}\widehat{T}^{p(j)} + \frac{1}{9}\widehat{T}^{p(j+1)} + \dots \approx_w \frac{1}{2}I + \frac{1}{2}\Theta$$

for most j. This implies the disjointness of the convolutions. Is the supply of weak limits of number-theoretic origin sufficient to establish that the spectrum of symmetric tensor powers of the operator $\widehat{T}$ is simple?

6. Semigroups of weak limits of rigid transformations. Let T be a transformation, let $\mathrm{WLim}(T)$ be the weak closure of the action $\{\widehat{T}^n : n \in \mathbb{Z}\}$, and let $\mathrm{ULim}(T)$ be the maximal unitary group in $\mathrm{WLim}(T)$. We know examples of nonmixing systems such that $\mathrm{WLim}(T)/\,\mathrm{ULim}(T)$ admits an explicit description; for example, it consists of the classes P^n, $n > 0$, where $P = \frac{1}{2}(I + \widehat{T})$ or $P = \frac{1}{2}(I + \Theta)$. However, there are no such descriptions for rigid weakly mixing rank one transformations. In the del Junco–Rudolph construction, the semigroup $\mathrm{WLim}(T)$ contains the operators $aI + (1-a)\widehat{T}$ and Θ. Do all of their possible products give a complete description of the semigroup $\mathrm{WLim}(T)/\,\mathrm{ULim}(T)$? The rigid rank one transformation defined by the parameters

$$\bar{s}_j = (1, 0, 1, 0, \dots, 1, 0, 1, 0), \quad r_j \to \infty,$$

is a good candidate for studying $\mathrm{WLim}(T)/\,\mathrm{ULim}(T)$.

The $n!$-*construction* given by the parameters $h_j = j!$, $R_j = j$, and $s_j(i) = (j-1)!$ for $j > 1$ is another candidate for research. The weak closure of such an infinite action T contains operators of the form $a\widehat{S}$, where $a \in [0,1]$ and $\widehat{S} \in \mathrm{ULim}(T)$. Are there others?

Exotic weak closure. Associated with a hypothetical orthogonal operator V such that $V^{n_i} \to I$ along some sequence $n_i \to \infty$ and the weak closure consists of some group of orthogonal operators and the zero operator is a Gaussian automorphism G with weak closure $\mathrm{WLim}(G) = \mathrm{ULim}(G) \cup \{\Theta\}$ such that $\mathrm{ULim}(G)$ is a continual group. Could a rigid rank one transformation have the same property?

Added in proof. J.-P. Touvenot informed the author about the impossibility of exotic examples: a rigid weakly mixing action always has a weak limit, which is not unitary and is not equal to Θ.

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