Dirichlet problem for the HJ PDE in the space of probability measures revisited

Yurii Averboukh,
(joint work with Ekaterina Kolpakova)

HSE University & Krasovskii Institute of mathematics and Mechanics
ayv@imm.uran.ru

November 13, 2026

Dirichlet problem in the space of probability measures

Let G be an open subset of the set of probability measures.

Dirichlet problem:

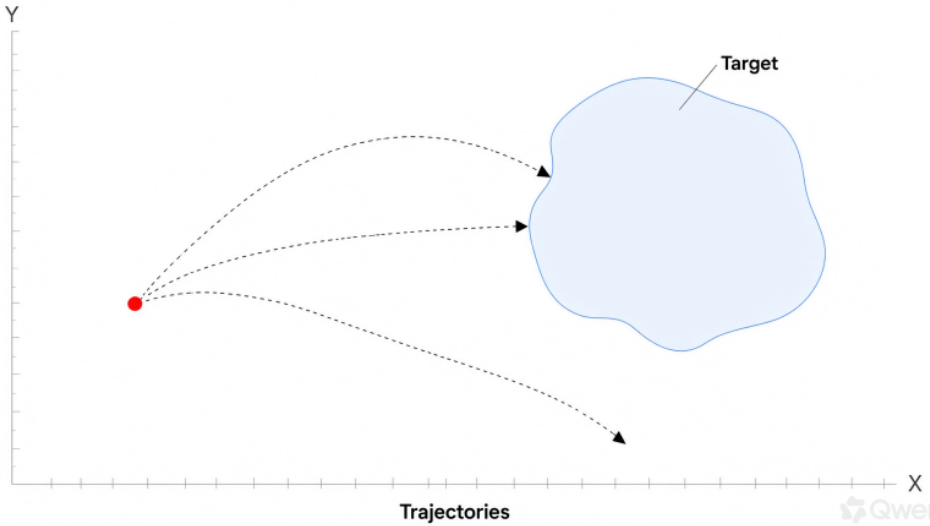
$$H(m, \nabla\phi(m)) + 1 - \phi(m) = 0,\ m \in G;$$

$$\phi(m) = 0,\ m \in \partial G.$$

Plan

1. Finite dimensional case
2. Wasserstein space
3. Time optimal problem in the Wasserstein space
4. Value function and its properties
5. Viscosity solution of the Dirichlet problem
6. Comparison principle
7. Characterization of the value function
8. Topological properties and example

Finite dimensional case. Time optimal problem



Finite dimensional case. Time optimal problem

- dynamics:

$$\frac{d}{dt}x(t) = f(x(t), u(t)),\ t \in [0, +\infty),\ x(t) \in \mathbb{R}^d,\ u(t) \in U;$$

- aim is to steer the system onto the set M;
- $G \triangleq \mathbb{R}^d \setminus M$;
- functional:

$$\tau(x(\cdot)) \triangleq \min\left\{t \text{ s.t. } x(t) \in M\right\};$$

- value function:

$$\mathrm{Val}(y) \triangleq \min\left\{\tau(x(\cdot)) \text{ s.t. } x(0) = y,\ (x(\cdot), u(\cdot)) \text{ obeys the dynamics, } u(t) \in U\right\}.$$

Dynamic programming and Kruzhkov transform

- Dynamic programming: for each $h \in [0, \mathrm{Val}(y)]$,

$$\mathrm{Val}(y) - h = \inf_{u(\cdot)} \mathrm{Val}(x(h, y, u(\cdot))),$$

where

$$\frac{d}{dt} x(t, y, u(\cdot)) = f(x(t, y, u(\cdot)), u(t)) \, x(0, y, u(\cdot)) = y.$$

- Kruzhkov transform:

$$\phi(x) = 1 - e^{-\mathrm{Val}(x)}.$$

- Equivalent form of the dynamic programming: for each $h \in [0, \mathrm{Val}(y)]$,

$$(1 - \phi(y))e^h = \sup_{u(\cdot)} \left[1 - \phi(x(h, y, u(\cdot)))\right].$$

Hamiltonian and the Dirichlet problem

- Hamiltonian: for $x \in G$, $p \in \mathbb{R}^d$:

$$H(x,p) \triangleq \min_{u \in U} \langle p, f(x,u) \rangle;$$

- Passing to the limit in

$$(1-\phi(y))e^h = \sup_{u(\cdot)} \left[1 - \phi(x(h,y,u(\cdot)))\right]$$

when $h \downarrow 0$, we get Dirichlet problem:

$$H(x, \nabla\phi(x)) + 1 - \phi(x) = 0,\ x \in G;$$

$$\phi(x) = 0,\ x \in \partial G.$$

Viscosity solution. Nonsmooth analysis

Let $x \in \mathbb{R}^d$, $r > 0$, $\phi : \mathbb{B}_r(x) \to \mathbb{R}$.

- subdifferential:

$$\partial^- \phi(x) \triangleq \left\{ p \in \mathbb{R}^d \text{ s.t. for each } v \liminf_{h \to 0, v' \to v} \frac{1}{h} \big[\phi(x + hv') - \phi(x) \big] \geq \langle p, v \rangle \right\};$$

- superdifferential:

$$\partial^+ \phi(x) \triangleq \left\{ p \in \mathbb{R}^d \text{ s.t. for each } v \limsup_{h \to 0, v' \to v} \frac{1}{h} \big[\phi(x + hv') - \phi(x) \big] \leq \langle p, v \rangle \right\}.$$

Viscosity solution [Subbotin, 1995]

- a bounded lower semicontinuous function $\phi : G \to \mathbb{R}$ is a supersolution of the HJ PDE provided that for each $x \in G$ and $p \in \partial^- \phi(x)$,

$$H(x, p) + 1 - \phi(x) \leq 0;$$

- a bounded upper semicontinuous function $\phi : G \to \mathbb{R}$ is a subsolution provided that for each $x \in G$ and $p \in \partial^+ \phi(x)$,

$$H(x, p) + 1 - \phi(x) \geq 0;$$

- ϕ is a subsolution (resp. supersolution) of the Dirichlet problem for the HJ PDE if it is a a subsolution (resp. supersolution) of the HJ PDE and meets the condition $\phi(x) = 0$, $x \in \partial G$.

Viscosity solution [Subbotin, 1995]

A bounded lower semicontinuos function $\phi : \mathrm{cl}\, G \to \mathbb{R}$ is called a viscosity solution the Dirichlet problem for HJ PDE if

- ϕ is a supersolution of problem the Dirichlet problem for HJ PDE;
- there exists a sequence of functions $\{\phi_k\}_{k=1}^{\infty}$ defined on $\mathrm{cl}\, G$ with values in $\mathbb{R}$ such that each function ϕ_k is a subsolution the Dirichlet problem for HJ PDE, one has that

$$\phi(x) = \lim_{k \to \infty} \phi_k(x).$$

Properties [Subbotin, 1995]

- the function ϕ defined by the rule: $\phi(x) = 1 - e^{-\mathrm{Val}(x)}$ is a solution of Dirichlet problem for HJ PDE;
- the solution of Dirichlet problem for HJ PDE is unique.

Comparison principle

If

- ϕ_1 is a subsolution of HJ PDE;
- ϕ_2 is a supersolution of HJ PDE;
- $\phi_1 \leq \phi_2$ on ∂G,

then $\phi_1 \leq \phi_2$ on G.

Proof via [Bardi and Capuzzo-Dolcetta, 1997]

- Assume the converse, i.e., let $\hat{x} \in G$ be s.t.

$$\phi_1(\hat{x}) > \phi_2(\hat{x}).$$

- Consider a double variable test function:

$$\Phi_\varepsilon(x_1, x_2) \triangleq \phi_1(x_1) - \phi_2(x_2) - \frac{1}{2\varepsilon}|x_1 - x_2|^2 - \beta(1 + |x_1|^2)^{\kappa/2} - \beta(1 + |x_2|^2)^{\kappa/2},$$

where $\kappa = 1 \wedge 1/C_f$, C_f satisfies $|f(x,u)| \leq C_f(1 + |x|^2)^{1/2}$, β is s.t.

$$\Phi_\varepsilon(\hat{x}, \hat{x}) = \phi_1(\hat{x}) - \phi_2(\hat{x}) - \beta(1 + |\hat{x}|^2)^{\kappa/2} - \beta(1 + |\hat{x}|^2)^{\kappa/2} =: \varkappa > 0.$$

- There exists $R > 0$ s.t. $\Phi_\varepsilon(x_1, x_2) < 0$ if $|x_1| \vee |x_2| \geq R$.

Proof II

- Let $(x_1^\varepsilon, x_2^\varepsilon)$ be a maximizer of Φ_ε. Notice that $|x_1^\varepsilon|, |x_2^\varepsilon| \leq R$.
- Obvious estimate:

$$\begin{aligned}
&\Phi_\varepsilon(x_1^\varepsilon, x_2^\varepsilon) \\
&\quad = \phi_1(x_1^\varepsilon) - \phi_2(x_2^\varepsilon) - \frac{1}{2\varepsilon}|x_1^\varepsilon - x_2^\varepsilon|^2 - \beta(1 + |x_1^\varepsilon|^2)^{\kappa/2} - \beta(1 + |x_2^\varepsilon|^2)^{\kappa/2} \\
&\quad \geq \Phi_\varepsilon(\hat{x}, \hat{x}) = \varkappa > 0.
\end{aligned}$$

- Claim: $|x_1^\varepsilon - x_2^\varepsilon| \to 0$ as $\varepsilon \to 0$.

Proof III

Claim: $(2\varepsilon)^{-1}|x_1^\varepsilon - x_2^\varepsilon|^2 \to 0$ as $\varepsilon \to 0$.

- Let $A \triangleq \sup_{|x|\le R}\left[\phi_1(x) - \phi_2(x) - 2\beta(1+|x|^2)^{\kappa/2}\right]$.
- Since $(x_1^\varepsilon, x_2^\varepsilon)$ is a maximizer,

$$A \le \Phi_\varepsilon(x_1^\varepsilon, x_2^\varepsilon) \le \phi_1(x_1^\varepsilon) - \phi_2(x_2^\varepsilon) - \beta(1+|x_1^\varepsilon|^2)^{\kappa/2} - \beta(1+|x_2^\varepsilon|^2)^{\kappa/2}.$$

- Use the assumptions that ϕ_1 is u.s.c., ϕ_2 is l.s.c. and the convergence $|x_1^\varepsilon - x_2^\varepsilon|^2 \to 0$ as $\varepsilon \to 0$:

$$\limsup_{\varepsilon\to 0}\left[\phi_1(x_1^\varepsilon) - \phi_2(x_2^\varepsilon) - \beta(1+|x_1^\varepsilon|^2)^{\kappa/2} - \beta(1+|x_2^\varepsilon|^2)^{\kappa/2}\right] \le A.$$

- Thus, $\frac{1}{2\varepsilon}|x_1^\varepsilon - x_2^\varepsilon|^2 \to 0$ as $\varepsilon \to 0$.

Proof IV. Dirichlet condition works

Case I: there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ converging to zero such that $(x_1^0, x_2^0) = \lim_{k\to\infty}(x_1^{\varepsilon_k}, x_2^{\varepsilon_k}) \in \partial(G \times G)$.

- For k one has that

$$\phi_1(x_1^{\varepsilon_k}) - \phi_2(x_2^{\varepsilon_k}) - \frac{1}{2\varepsilon_k}|x_1^{\varepsilon_k} - x_2^{\varepsilon_k}|^2 - \beta(1 + |x_1^{\varepsilon_k}|^2)^{\kappa/2} - \beta(1 + |x_2^{\varepsilon_k}|^2)^{\kappa/2} \geq \varkappa.$$

- Pass to the limit as $k \to \infty$, use the boundary condition and semicontinuity of the function ϕ_1, ϕ_2:

$$-\beta(1 + |x_1^0|^2)^{\kappa/2} - \beta(1 + |x_1^0|^2)^{\kappa/2} \geq \varkappa > 0.$$

Proof V. HJ PDE works

Case II: for every sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ converging to zero such that $(x_1^{\varepsilon_k}, x_2^{\varepsilon_k}) \to (x_1^0, x_2^0)$ as $k \to \infty$ one has that $(x_1^0, x_2^0) \in \mathrm{int}(G \times G)$.

Auxiliary functions:

$$\begin{aligned}\psi_{1,\varepsilon}(x_1) &\triangleq \Phi_\varepsilon(x_1, x_2^\varepsilon) \\ &= \phi_1(x_1) - \phi_2(x_2^\varepsilon) - \frac{1}{2\varepsilon}|x_1 - x_2^\varepsilon|^2 - \beta(1 + |x_1|^2)^{\kappa/2} - \beta(1 + |x_2^\varepsilon|^2)^{\kappa/2};\end{aligned}$$

$$\begin{aligned}\psi_{2,\varepsilon}(x_2) &\triangleq -\Phi_\varepsilon(x_1^\varepsilon, x_2) \\ &= -\phi_1(x_1^\varepsilon) + \phi_2(x_2) + \frac{1}{2\varepsilon}|x_1^\varepsilon - x_2|^2 + \beta(1 + |x_1^\varepsilon|^2)^{\kappa/2} + \beta(1 + |x_2|^2)^{\kappa/2}.\end{aligned}$$

Proof VI. HJ PDE works

- $\psi_{1,\varepsilon}$ attains a maximum at x_1^ε. Thus, $0 \in \partial^+ \psi_{1,\varepsilon}(x_1^\varepsilon)$, i.e.,

$$\varepsilon^{-1}(x_1^\varepsilon - x_2^\varepsilon) + \beta\kappa\big[(1+|x_1^\varepsilon|^2)^{1/2}\big]^{\kappa-1}\frac{x_1^\varepsilon}{(1+|x_1^\varepsilon|^2)^{1/2}} =: s_1^\varepsilon \in \partial^+\phi_1(x_1^\varepsilon).$$

- $\psi_{2,\varepsilon}$ attains a minimum at x_2^ε. Thus,

$$\varepsilon^{-1}(x_1^\varepsilon - x_2^\varepsilon) - \beta\kappa\big[(1+|x_2^\varepsilon|^2)^{1/2}\big]^{\kappa-1}\frac{x_2^\varepsilon}{(1+|x_2^\varepsilon|^2)^{1/2}} =: s_2^\varepsilon \in \partial^-\phi_2(x_2^\varepsilon).$$

Proof VII. HJ PDE works

- ϕ_1 is a viscosity subsolution:

$$H(x_1^\varepsilon, s_1^\varepsilon) + 1 - \phi_1(x_1^\varepsilon) \geq 0.$$

- ϕ_2 is a viscosity supersolution:

$$H(x_2^\varepsilon, s_2^\varepsilon) + 1 - \phi_2(x_2^\varepsilon) \leq 0.$$

- $H(x_1^\varepsilon, s_1^\varepsilon) - H(x_2^\varepsilon, s_2^\varepsilon) \geq \phi_1(x_1^\varepsilon) - \phi_2(x_2^\varepsilon)$.

Proof VIII. Estimate of the Hamiltonians

- Notice that

$$\left| \left[(1 + |x_i^\varepsilon|^2)^{1/2} \right]^{\kappa - 1} \frac{x_i^\varepsilon}{(1 + |x_i^\varepsilon|^2)^{1/2}} \right| \leq \left[(1 + |x_i^\varepsilon|^2)^{1/2} \right]^{\kappa - 1}.$$

- Since $\kappa = 1 \wedge (C_f)^{-1}$, where C_f is s.t. $|f(x, u)| \leq C_f (1 + |x|^2)^{1/2}$,

$$\left\langle \kappa \left[(1 + |x_i^\varepsilon|^2)^{1/2} \right]^{\kappa - 1} \frac{x_i^\varepsilon}{(1 + |x_i^\varepsilon|^2)^{1/2}}, f(x_i, u) \right\rangle \leq (1 + |x_i^\varepsilon|^2)^{\kappa/2}.$$

Proof IX. Estimate of the Hamiltonians

- Let u_ε be s.t. $H(x_2^\varepsilon, s_2^\varepsilon) = \langle s_2^\varepsilon, f(x_2^\varepsilon, u_\varepsilon)\rangle$.
- The definition of s_i^ε gives that

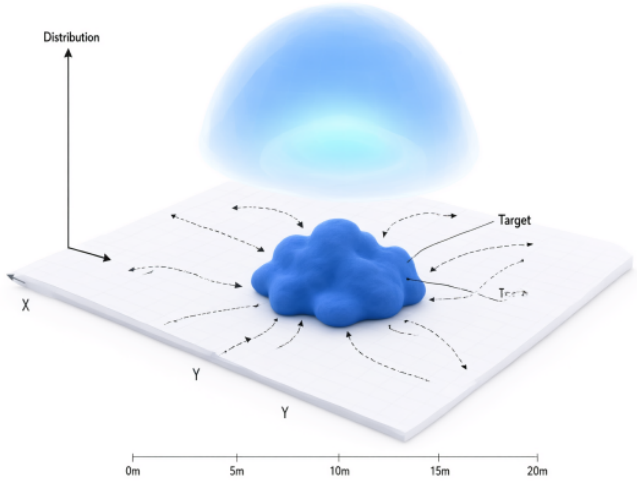
$$\begin{aligned} &H(x_1^\varepsilon,s_1^\varepsilon) - H(x_2^\varepsilon, s_2^\varepsilon) \\ &\quad\le \varepsilon^{-1}\langle (x_1^\varepsilon - x_2^\varepsilon), f(x_1^\varepsilon, u_\varepsilon) - f(x_2^\varepsilon, u_\varepsilon)\rangle + \beta(1+|x_1^\varepsilon|^2)^{\kappa/2} + \beta(1+|x_2^\varepsilon|^2)^{\kappa/2} \\ &\quad\le \varepsilon^{-1}L|x_1^\varepsilon - x_2^\varepsilon|^2 + \beta(1+|x_1^\varepsilon|^2)^{\kappa/2} + \beta(1+|x_2^\varepsilon|^2)^{\kappa/2}, \end{aligned}$$

  where L is a Lischitz constant.
- Thus,

$$(2L+1)\frac{1}{2\varepsilon}|x_1^\varepsilon - x_2^\varepsilon|^2 \ge \Phi_\varepsilon(x_1^\varepsilon, x_2^\varepsilon) \ge \varkappa > 0.$$

- Passing to the limit, we obtain the contadiction.

Space of probability case. Time optimal problem



Qwen

Natural projection

If $X_1, \ldots, X_n$ are sets, $i_1 \ldots, i_k \in \{1, \ldots, n\}$, then
$\mathrm{p}^{i_1,\ldots,i_k} : X_1 \times \ldots \times X_n \to X_{i_1} \times \ldots \times X_{i_k}$ is defined by the rule

$$\mathrm{p}^{i_1,\ldots,i_k}(x_1, \ldots, x_n) \triangleq (x_{i_1}, \ldots, x_{i_k}).$$

Measurable space

Let Ω be a set. $\mathcal{F} \subset 2^{\Omega}$ is called a σ-algebra on Ω provided that

- $\Omega \in \mathcal{F}$;
- if $A, B \in \mathcal{F}$, then
$$A \cap B,\ A \cup B,\ A \setminus B \ \in \mathcal{F};$$
- if $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$, then
$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

A set Ω and a σ-algebra $\mathcal{F}$ on it form a measurable space.

Example. Borel σ-algebra

Let (X, ρ) be a Polish space. The Borel σ-algebra on X is the minimal σ-algebra containing all open set. It is denoted by $\mathcal{B}(X)$.

Measures and probabilities

Let $(\Omega, \mathcal{F})$ be a measurable space.
A function $m : \mathcal{F} \to [0, +\infty]$ is called a measure provided that, for every $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$, such that $A_i \cap A_j = \varnothing$ whenever $i \neq j$, one has that

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i).$$

A measure m is called a probability if

$$m(\Omega) = 1.$$

Borel measures and Borel probabilities

Let (X, ρ) be a Polish space.
Each measure defined on $\mathcal{B}(X)$ is called a Borel measure, a probability defined on $\mathcal{B}(X)$ is called a Borel probability.

- $\mathcal{M}(X)$ denotes the set of Borel measures on X;
- $\mathcal{M}_f(X)$ denotes the set of finite Borel measures on X;
- $\mathcal{P}(X)$ denotes the set of Borel probabilities on X.

$$\mathcal{P}(X) \subset \mathcal{M}_f(X) \subset \mathcal{M}(X).$$

Push-forward measure

Let

- $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ be measurable spaces;
- m be a measure on $\mathcal{F}$;
- $g : \Omega \to \Omega'$ be a $\mathcal{F}/\mathcal{F}'$-measurable function.

Then $g\sharp m$ is a measure on $\mathcal{F}'$ defined by the rule: for each $\Upsilon \in \mathcal{F}'$,

$$(g\sharp m)(\Upsilon) \triangleq m\big(g^{-1}(\Upsilon)\big).$$

Narrow convergence

A sequence $\{m_n\}_{n=1}^{\infty} \subset \mathcal{M}_f(X)$ narrowly converges to $m \in \mathcal{M}_f(X)$ if, for every $\phi \in C_b(X)$,

$$\int_X \phi(x) m_n(dx) \to \int_X \phi(x) m(dx) \text{ as } n \to \infty.$$

The space $\mathcal{P}(X)$ is closed w.r.t. the narrow convergence.

Wasserstein space

Let (X, ρ) be a Polish space.
The space $\mathcal{P}^2(X)$ contains all probabilities $m \in \mathcal{P}(X)$ such that, for some $x_* \in X$,

$$\int_X \rho^2(x, x_*) m(dx) < \infty.$$

If X is a Banach space, then we denote

$$\varsigma(m) \triangleq \left[\int_X \|x\|^2 m(dx) \right]^{1/2}.$$

Wassertein distance

Let $m_1, m_2 \in \mathcal{P}^2(X)$. Then,

$$W_2(m_1, m_2) = \left[\inf_{\pi \in \Pi(m_1, m_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho(x_1, x_2)^2 \pi(d(x_1, x_2)) \right]^{1/2}.$$

Here $\Pi(m_1, m_2)$ stands for the set of probabilities $\pi \in \mathcal{P}^2(X \times X)$ such that $\mathrm{p}^i \sharp \pi = m_i$.

Space of functions

Let $m \in \mathcal{P}(\mathbb{R}^d)$, then

$$L^p(m) \triangleq \left\{ \varphi : \mathbb{R}^d \to \mathbb{R}^k \text{ s.t. } \int_{\mathbb{R}^d} |\varphi(x)|^p m(dx) < \infty \right\}.$$

If φ_1, φ_2 are such that $x \mapsto \langle \varphi(x), \varphi_2(x) \rangle \in L^1(m)$, then

$$\langle \varphi_1, \varphi_2 \rangle_m \triangleq \int_{\mathbb{R}^d} \langle \varphi_1(x), \varphi_2(x) \rangle m(dx).$$

Relaxed controls on a finite time interval

Let U be a metric compact, $T > 0$.

$$\mathcal{U}_T \triangleq \{\xi_T \in \mathcal{M}_f([0,T] \times U) \text{ s.t. } \mathrm{p}^1 \sharp \xi_T = \lambda\}.$$

Here λ stands for the Lebesgue measure.

If $\xi_T \in \mathcal{U}_T$, then there exists a weakly measurable family $(\xi_T(\cdot|t))_{t\in[0,T]} \subset \mathcal{P}(U)$ s.t., for every $\varphi \in C([0,T] \times U)$,

$$\int_0^T \int_U \varphi(t,u)\xi_T(du|t)dt = \int_{[0,T]\times U} \varphi(d(t,u))\xi_T(d(t,u)).$$

Relaxed controls on the infinite time interval

Let U be a metric compact.

$$\mathcal{U} \triangleq \{\xi \in \mathcal{M}([0,+\infty) \times U) \text{ s.t. } \mathrm{p}^1 \sharp \xi = \lambda\}.$$

If $\xi \in \mathcal{U}$, $T > 0$, then ξ_T denotes the restriction of ξ on $[0,T] \times U$.

A sequence $\{\xi^n\}_{n=1}^{\infty} \subset \mathcal{U}$ converges to $\xi \in \mathcal{U}$ provided that, for every $T > 0$ (equivalently, every natural T), $\{\xi_T^n\}_{n=1}^{\infty} \subset \mathcal{U}$ converges narrowly to $\xi_T \in \mathcal{U}$.

The space $\mathcal{U}$ is compact.

Relaxed controls vs measurable controls

Let $u(\cdot) : [0, +\infty) \to U$, then $\xi_{u(\cdot)}$ s.t.

$$\xi_{u(\cdot)}(du|t) \triangleq \delta_{u(t)}(du)$$

lies in $\mathcal{U}$.
This embedding is dense in $\mathcal{U}$.

Dynamics

$$\partial_t m(t) + \operatorname{div}(f(x, m(t), u(t))m(t)) = 0, \quad t \in [0, +\infty).$$

Here,

- $x \in \mathbb{R}^d$,
- $m(t) \in \mathcal{P}^2(\mathbb{R}^d)$,
- $u(t) \in U$, the set U is a metric compact.

Assumptions

- The function f is continuous in all arguments.
- There exists a constant $L > 0$ such that, for every $x_1, x_2 \in \mathbb{R}^d$, $m_1, m_2 \in \mathcal{P}^2(\mathbb{R}^d)$, $u \in U$, one has the inequality

$$|f(x_1, m_1, u) - f(x_2, m_2, u)| \leq L(|x_1 - x_2| + W_2(m_1, m_2)).$$

- There exists a constant $C_f > 0$ such that, for every $x \in \mathbb{R}^d$, $m \in \mathcal{P}^2(\mathbb{R}^d)$, $u \in U$,

$$|f(x, m, u)| \leq C_f \left(\mathscr{a}(x) + \int_{\mathbb{R}^d} \mathscr{a}(x) m(dx) \right),$$

where

$$\mathscr{a}(x) \triangleq (1 + |x|^2)^{1/2}.$$

Motion

Let $\xi \in \mathcal{U}$. We consider the continuity equation

$$\partial_t m(t) + \operatorname{div}\left(\int_U f(x, m(t), u)\xi(du|t) \cdot m(t) \right) = 0, \quad m(0) = \mu.$$

A function $m(\cdot) : [0, +\infty) \mapsto \mathcal{P}^2(\mathbb{R}^d)$ is a motion on $[0, +\infty)$ provided that, for every $\varphi \in C_c^\infty((0, +\infty) \times \mathbb{R}^d)$, one has that

$$\int_{[0,\infty)\times U} \int_{\mathbb{R}^d} \partial_t \varphi(t, x) + \langle \nabla \varphi(t, x), f(x, m(t), u) \rangle \, m(t, dx)\xi(d(t, u)) = 0.$$

A motion satisfying $m(0) = \mu$ is denoted by $m(\cdot; \mu, \xi)$.

Particle representation

Given $\xi \in \mathcal{U}$, a measure-valued function $m(\cdot) \in C([0,+\infty); \mathcal{P}^2(\mathbb{R}^d))$ is a motion if and only if there exists a probability $\chi \in \mathcal{P}(C([0,+\infty); \mathbb{R}^d))$ such that the following properties holds true:

- $m(t) \triangleq e_t \sharp \chi$;
- χ-a.e. curves $x(\cdot)$ are such that

$$\frac{d}{dt}x(\cdot) = \int_U f(x(t), m(t), u)\xi(du|t).$$

This property directly follows from the seminal superposition principle [Ambrosio et al., 2008, Theorem 8.2.1].

Time optimal problem

Let M be a closed subset of $\mathcal{P}^2(\mathbb{R}^d)$;

$$G \triangleq \mathcal{P}^2(\mathbb{R}^d) \setminus M.$$

Define $\tau : C([0, +\infty); \mathcal{P}^2(\mathbb{R}^d)) \to [0, +\infty]$ by the rule:

$$\tau(m(\cdot)) \triangleq \inf\{t \in [0; +\infty) : \ m(t) \in M\}.$$

Value function

$$\mathrm{Val}(\mu) = \inf\{\tau(m(\cdot;\mu,\xi)) : \quad \xi \in \mathcal{U}\}.$$

Properties of the value function

- If $\mu \in \operatorname{cl} G$ is such that $\operatorname{Val}(\mu) < +\infty$. Then, there exists a relaxed control $\xi_* \in \mathcal{U}$ such that
$$\operatorname{Val}(\mu) = \tau(m(\cdot; \mu, \xi_*)).$$
- The function Val is lower semicontinuous.

Dynamic programming

Theorem

For every $\mu \in \mathrm{cl}\, G$ *and* $h \in [0, \mathrm{Val}(\mu)]$,

$$\mathrm{Val}(\mu) - h = \inf_{\xi \in \mathcal{U}_h} \mathrm{Val}(m(h; \mu, \xi)).$$

Dirichlet problem

- What is Hamiltonian?
- How one should define the viscosity solution?

Highlights form finite dimensional case

- key idea: add the regularization terms: $-\beta(1+|x_1|^2)^{\kappa/2} - \beta(1+|x_2|^2)^{\kappa/2}$;
- naive regularization term for the Wasserstein space:

$$-\beta\left[\int_{\mathbb{R}^d} |x|^2 m_1(dx)\right]^{\kappa} - \beta\left[\int_{\mathbb{R}^d} |x|^2 m_2(dx)\right]^{\kappa};$$

- issue: the set

$$\left\{ m \text{ s.t. } \int_{\mathbb{R}^d} |x|^2 m(dx) \leq C \right\}$$

  is not compact.

Nonsmooth analysis in the Wasserstein space. Tangents

Let $m \in \mathcal{P}^2(\mathbb{R}^d)$.

- distributed tangent [Ambrosio et al., 2008, Jimenez et al., 2020]: a probability measure $\gamma(dxdv)$ s.t. $\gamma \in \mathcal{P}^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $\mathrm{p}^1 \sharp \gamma = m$;
- baricentric tangent [Ambrosio et al., 2008, Badreddine and Frankowska, 2021]: a function $F \in L^2(m)$;
- sublinear tangent: a Borel function $F : \mathbb{R}^d \to \mathbb{R}^d$ s.t. for some $C > 0$ $|F(x)| \leq C\mathscr{a}(x)$ m-a.e.

Space of sublinear tangents

- SL(m) consists of all Borel function $F : \mathbb{R}^d \to \mathbb{R}^d$ s.t., for some constant $C > 0$, and m-a.e. $x \in \mathbb{R}^d$,
$$|F(x)| \leq C\mathscr{a}(x).$$
- convergence on SL(m): a sequence of functions $\{F_n\}_{n=1}^{\infty} \subset \mathrm{SL}(m)$ converges to $F \in \mathrm{SL}(m)$ provided that
  - there exists a constant C such that
  $$|F_n(x)| \leq C\mathscr{a}(x) \text{ for } m\text{-a.e. } x \in \mathbb{R}^d;$$
  - $F_n(x) \to F(x)$ as $n \to \infty$ for m-a.e. $x \in \mathbb{R}^d$.

Dual space

- $\mathrm{LD}(m) = L^1_{\mathscr{a}(x)}$ consists of all functions $s : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |s(x)| \mathscr{a}(x) m(dx) < \infty.$$

- For every $F \in \mathrm{SL}(m)$, $s \in \mathrm{LD}(m)$, $\langle s, F \rangle_m$ is well-defined.
- If $\{F_n\}_{n=1}^{\infty} \subset \mathrm{SL}(m)$ converges to $F \in \mathrm{SL}(m)$, then $\langle s, F_n \rangle_m \to \langle s, F \rangle_m$.

Hamiltonian in the Wasserstein space

For $m \in \mathcal{P}^2(\mathbb{R}^d)$ and $s \in \mathrm{LD}(m)$,

$$H(m, s) := \inf_{u \in U} \int_{\mathbb{R}^d} \langle s(x), f(x, m, u) \rangle \, m(dx).$$

Dirichlet problem in the Wasserstein space

$$H(m, \nabla\phi(m)) + 1 - \phi(m) = 0,\ m \in G;$$
$$\phi(m) = 0,\ m \in \partial G.$$

Subdifferential

Let $m \in \mathcal{P}^2(\mathbb{R}^d)$, $r > 0$, and let ψ be a function defined on $\mathbb{B}_r(m)$ with values in $\mathbb{R} \cup \{+\infty, -\infty\}$ such that $\psi(m) \in \mathbb{R}$.

Definition

A subdifferential of the function ψ at the measure m consists of all functions $s \in \mathrm{LD}(m)$ such that, for every $F \in \mathrm{SL}(m)$,

$$\liminf_{h \downarrow 0, F' \in \mathrm{SL}(m),\, F' \to F} \frac{\psi((\mathrm{Id} + hF')\sharp m) - \psi(m) - h\langle s, F\rangle_m}{h} \geq 0.$$

The subdifferential of the function ψ at the measure m is denoted by $\partial^- \psi(m)$.

Superdifferential

Let $m \in \mathcal{P}^2(\mathbb{R}^d)$, $r > 0$, and let ψ be a function defined on $\mathbb{B}_r(m)$ with values in $\mathbb{R} \cup \{+\infty, -\infty\}$ such that $\psi(m) \in \mathbb{R}$.

Definition

A superdifferential of the function ψ at the measure m consists of all functions $s \in \mathrm{LD}(m)$ such that, for every $F \in \mathrm{SL}(m)$,

$$\limsup_{h \downarrow 0, F' \in \mathrm{SL}(m),\, F' \to F} \frac{\psi((\mathrm{Id} + hF')\sharp m) - \psi(m) - h\langle s, F\rangle_m}{h} \leq 0.$$

The superdifferential of the function ψ at the measure m is denoted by $\partial^+\psi(m)$.

Viscosity supersolution

- A bounded lower semicontinuous function $\phi : \mathrm{cl}\, G \to \mathbb{R}$ is called a viscosity supersolution of HJ PDE provided that, for every $m \in G$ and $s \in \partial^- \phi(m)$,

$$H(m, s) + 1 - \phi(m) \leq 0.$$

- A bounded lower semicontinuous function $\phi : \mathrm{cl}\, G \to \mathbb{R}$ is called a viscosity supersolution of Dirichlet problem for HJ PDE if it is a viscosity supersolution of HJ PDE and satisfies $\phi(m) = 0$ for $m \in \partial G$.

Viscosity subsolution

- A bounded upper semicontinuous function $\phi : \mathrm{cl}\, G \to \mathbb{R}$ is called a viscosity subsolution of HJ PDE provided that, for every $m \in G$ and $s \in \partial^+ \phi(m)$,

$$H(m, s) + 1 - \phi(m) \geq 0.$$

- A bounded upper semicontinuous function $\phi : \mathrm{cl}\, G \to \mathbb{R}$ is called a viscosity subsolution of Dirichlet problem for HJ PDE if it is a viscosity subsolution of HJ PDE and satisfies $\phi(m) = 0$ for $m \in \partial G$.

Viscosity solution

A bounded lower semicontinuos function $\phi : \mathrm{cl}\, G \to \mathbb{R}$ is called a viscosity solution of the Dirichlet problem for HJ PDE if

- ϕ is a supersolution of problem of the Dirichlet problem
- there exists a sequence of functions $\{\phi_k\}_{k=1}^{\infty}$ defined on $\mathrm{cl}\, G$ with values in $\mathbb{R}$ such that each function ϕ_k is a subsolution of the Dirichlet problem while, for every $m \in G$, one has that

$$\phi(m) = \lim_{k \to \infty} \phi_k(m).$$

Comparison principle

Theorem

If

- ϕ_1 *is a subsolution of HJ PDE;*
- ϕ_2 *is a supersolution of HJ PDE;*
- $\phi_1 \leq \phi_2$ *on* ∂G,

then $\phi_1 \leq \phi_2$ *on* G.

Squared Wasserstein distance [Ambrosio et al., 2008, Theorem 10.2.2]

Let

- $m, m_b \in \mathcal{P}^2(\mathbb{R}^d)$,
- $\vartheta[m, m_b] \in \Pi(m, m_b)$ be an optimal plan between m and m_b;
- $\hat{\vartheta}[m, m_b] \in L^2(m) \subset \mathrm{LD}(m)$ be defined by the rule:

$$\hat{\vartheta}[m, m_b](x) \triangleq \int_{\mathbb{R}^d} (x - y)\vartheta[m, m_b](dy|x).$$

Then, $\hat{\vartheta}[m, m_b]$ is an element of the superdifferential of the function $m' \mapsto \frac{1}{2} W_2^2(m', m_b)$ at m.

Property

The set $\mathcal{K} \subset \mathcal{P}^2(\mathbb{R}^d)$ is compact iff there exists a function $g : [1, +\infty) \to [0, +\infty)$ s.t.

- g is increasing;
- $g(r) \to \infty$ as $r \to \infty$;
- $$\sup_{m \in \mathcal{K}} \int_{\mathbb{R}^d} g(a(x))a^2(x)m(dx) < \infty.$$

Moment gauge function

Lemma

Let $m \in \mathcal{P}^2(\mathbb{R}^d)$. Then, there exists a continuously differentiable function $g : (0, +\infty) \to (0, +\infty)$ such that

- *g is positive, monotonously increasing on $[1, +\infty)$ with $g(r) \to +\infty$ as $r \to +\infty$;*
- *$\int_{\mathbb{R}^d} a^2(x) g(a(x)) m(dx) < \infty$;*
- *$rg'(r) \leq g(r)$, where g' stands for the derivative of the function g.*

Moment gauge function. Construction I

- Since $m \in \mathcal{P}^2(\mathbb{R}^d)$,
$$\lim_{R\to\infty} \int_{\mathscr{a}(x)\geq R} \mathscr{a}^2(x) m(dx) \to 0.$$
- sequence $\{R_n\}_{n=0}^{\infty}$: $R_0 \triangleq 0$, $R_1 \triangleq 1$; for $n \geq 1$, we choose $R_{n+1} \geq (R_n + 1)$ s.t.
$$\int_{\mathscr{a}(x)\geq R_{n+1}} \mathscr{a}(x) m(dx) \leq (n+1)^{-1} 2^{-n}$$
and
$$nR_{n+1} \geq (n+2)R_n.$$

Moment gauge function. Construction II

An auxiliary function $g_0(\cdot)$ is defined by the rule:

- if $r \in [R_0, R_1]$, then $g_0(r) \triangleq 3$;
- if $r \in [R_n, R_{n+1}]$ for some $n \geq 1$, then we put

$$g_0(r) \triangleq (n+2) + (r - R_n)(R_{n+1} - R_n)^{-1}.$$

Key property:

$$\int_{\mathbb{R}^d} a^2(x) g_0(a(x)) m(dx) < \infty.$$

The function g is constructed from g_0 using so called rounding corners lemma.

Growth rate for the function g

Lemma

If a function $g : (0, +\infty) \to (0, +\infty)$ satisfies conditions of the previous lemma, then, for every $r \in (0, +\infty)$,

$$g(2r) \leq (1 - \ln 2)^{-1} g(r).$$

Potential energy function

Let $g : (0, +\infty) \to (0, +\infty)$ be such that

- g is positive, monotonously increasing on $[1, +\infty)$ with $g(r) \to +\infty$ as $r \to +\infty$;
- $rg'(r) \leq g(r)$, where g' stands for the derivative of the function g.

For $m \in \mathcal{P}^2(\mathbb{R}^d)$, denote

$$V_g(m) := \int_{\mathbb{R}^d} \mathscr{a}^2(x) g(\mathscr{a}(x)) m(dx).$$

The function $m \mapsto V_g(m)$ is l.s.c. [Ambrosio et al., 2008].

Potential energy function

Lemma

Let $g : (0, +\infty) \to (0, +\infty)$ *be such that*

- g *is positive, monotonously increasing on* $[1, +\infty)$ *with* $g(r) \to +\infty$ *as* $r \to +\infty$*;*
- $rg'(r) \leq g(r)$*, where* g' *stands for the derivative of the function* g*.*

Assume, additionally, that

$$V_g(m) := \int_{\mathbb{R}^d} a^2(x) g(a(x)) m(dx) < \infty.$$

Then, the function $\mathcal{J}_g(\cdot)$ *defined by the rule:*

$$\mathcal{J}_g(x) \triangleq \left[2g(a(x)) + a(x)g'(a(x))\right]x$$

lies in the intersection of $\partial^- V_g(m)$ *and* $\partial^+ V_g(m)$*.*

Proof of the comparison principle

- Assume the converse, i.e., let $\hat{m} \in G$ be s.t.

$$\phi_1(\hat{m}) > \phi_2(\hat{m}).$$

- Let g be a moment gauge function for the measure $\hat{m}$.
- Consider a double variable test function:

$$\Phi_\varepsilon(m_1, m_2) \triangleq \phi_1(m_1) - \phi_2(m_2) - \frac{1}{2\varepsilon} W_2^2(m_1, m_2) - \beta\big[V_g(m_1)\big]^\kappa - \beta\big[V_g(m_2)\big]^\kappa,$$

where $\kappa = 1 \wedge 1/(6C_f)$, β is s.t.

$$\Phi_\varepsilon(\hat{m}, \hat{m}) = \phi_1(\hat{m}) - \phi_2(\hat{m}) - 2\beta\big[V_g(\hat{m})\big]^\kappa =: \varkappa > 0.$$

- There exists $C_1 >$ s.t. $\Phi_\varepsilon(m_1, m_2) < 0$ if $V_g(m_1) \vee V_g(m_2) \geq C_1$.

Proof II

- The set $\mathcal{K} := \{m \in \mathrm{cl}(G) \text{ s.t. } V_g(m) \leq C_1\}$ is compact;
- The function $\mathcal{K}^2 \ni (m_1, m_2) \mapsto \Phi_\varepsilon(m_1, m_2)$ is u.s.c.
- Let $(m_1^\varepsilon, m_2^\varepsilon) \in \mathcal{K}^2$ be a maximizer of Φ_ε.
- Obvious estimate:

$$\begin{aligned}
&\Phi_\varepsilon(m_1^\varepsilon, m_2^\varepsilon) \\
&\quad = \phi_1(m_1^\varepsilon) - \phi_2(m_2^\varepsilon) - \frac{1}{2\varepsilon} W_2^2(m_1^\varepsilon, m_2^\varepsilon) - \beta\big[V_g(m_1^\varepsilon)\big]^\kappa - \beta\big[V_g(m_2^\varepsilon)\big]^\kappa \\
&\quad \geq \Phi_\varepsilon(\hat{m}, \hat{m}) = \varkappa > 0.
\end{aligned}$$

- Claim: $W_2(m_1^\varepsilon, m_2^\varepsilon)$ as $\varepsilon \to 0$.

Proof III

Claim: $(2\varepsilon)^{-1} W_2^2(m_1^\varepsilon, m_2^\varepsilon) \to 0$ as $\varepsilon \to 0$.

- Let $A \triangleq \sup_{|x|\le R} \left[\phi_1(x) - \phi_2(x) - 2\beta\left[V_g(m)\right]^\kappa\right]$.
- Since $(m_1^\varepsilon, m_2^\varepsilon)$ is a maximizer,

$$A \le \Phi_\varepsilon(m_1^\varepsilon, m_2^\varepsilon) \le \phi_1(m_1^\varepsilon) - \phi_2(m_2^\varepsilon) - \beta\left[V_g(m_1^\varepsilon)\right]^\kappa - \beta\left[V_g(m_2^\varepsilon)\right]^\kappa.$$

- Use the assumptions that ϕ_1 is u.s.c., ϕ_2 is l.s.c. and the convergence $W_2(m_1^\varepsilon, m_2^\varepsilon) \to 0$ as $\varepsilon \to 0$:

$$\limsup_{\varepsilon\to 0} \left[\phi_1(m_1^\varepsilon) - \phi_2(m_2^\varepsilon) - \beta\left[V_g(m_1^\varepsilon)\right]^\kappa - \beta\left[V_g(m_2^\varepsilon)\right]\right] \le A.$$

- Thus, $\frac{1}{2\varepsilon} W_2^2(m_1^\varepsilon, m_2^\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof IV. Dirichlet condition works

Case I: there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ converging to zero such that $(m_1^0, m_2^0) = \lim_{k\to\infty}(m_1^{\varepsilon_k}, m_2^{\varepsilon_k}) \in \partial(G \times G)$.

- For k one has that

$$\phi_1(m_1^{\varepsilon_k}) - \phi_2(m_2^{\varepsilon_k}) - \frac{1}{2\varepsilon_k} W_2^2(m_1^{\varepsilon_k}, m_2^{\varepsilon_k}) - \beta\big[V_g(m_1^{\varepsilon_k})\big]^{\kappa} - \beta\big[V_g(m_2^{\varepsilon_k})\big]^{\kappa} \geq \varkappa.$$

- Pass to the limit as $k \to \infty$, use the boundary condition and semicontinuity of the function ϕ_1, ϕ_2:

$$-\beta\big[V_g(m_1^0)\big]^{\kappa} - \beta\big[V_g(m_2^0)\big]^{\kappa} \geq \varkappa > 0.$$

Proof V. HJ PDE works

Case II: for every sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ converging to zero such that $(m_1^{\varepsilon_k}, m_2^{\varepsilon_k}) \to (m_1^0, m_2^0)$ as $k \to \infty$ one has that $(m_1^0, m_2^0) \in \mathrm{int}(G \times G)$.

Auxiliary functions:

$$\begin{aligned}\psi_{1,\varepsilon}(m_1) &\triangleq \Phi_\varepsilon(m_1, m_2^\varepsilon) \\ &= \phi_1(m_1) - \phi_2(m_2^\varepsilon) - \frac{1}{2\varepsilon} W_2^2(m_1, m_2^\varepsilon) - \beta\big[V_g(m_1)\big]^\kappa - \beta\big[V_g(m_2^\varepsilon)\big]^\kappa;\end{aligned}$$

$$\begin{aligned}\psi_{2,\varepsilon}(m_2) &\triangleq -\Phi_\varepsilon(m_1^\varepsilon, m_2) \\ &= -\phi_1(m_1^\varepsilon) + \phi_2(m_2) + \frac{1}{2\varepsilon} W_2^2(m_1^\varepsilon, m_2) + \beta\big[V_g(m_1^\varepsilon)\big]^\kappa + \beta\big[V_g(m_2)\big]^\kappa.\end{aligned}$$

Proof VI. HJ PDE works

- $\psi_{1,\varepsilon}$ attains a maximum at m_1^ε. Thus, $0 \in \partial^+ \psi_{1,\varepsilon}(m_1^\varepsilon)$, i.e.,

$$s_1^\varepsilon \triangleq \varepsilon^{-1} \hat{\vartheta}[m_1^\varepsilon, m_2^\varepsilon] + \kappa\beta \big[V_g(m_1^\varepsilon)\big]^{\kappa-1} \cdot j_g \in \partial^+ \phi_1(m_1^\varepsilon).$$

- $\psi_{2,\varepsilon}$ attains a minimum at m_2^ε. Thus,

$$s_2^\varepsilon \triangleq -\varepsilon^{-1} \hat{\vartheta}[m_2^\varepsilon, m_1^\varepsilon] - \kappa\beta \big[V_g(m_2^\varepsilon)\big]^{\kappa-1} \cdot j_g \in \partial^- \phi_2(m_2^\varepsilon).$$

Proof VII. HJ PDE works

- ϕ_1 is a viscosity subsolution:

$$H(m_1^\varepsilon, s_1^\varepsilon) + 1 - \phi_1(m_1^\varepsilon) \geq 0.$$

- ϕ_2 is a viscosity supersolution:

$$H(m_2^\varepsilon, s_2^\varepsilon) + 1 - \phi_2(m_2^\varepsilon) \leq 0.$$

- $H(m_1^\varepsilon, s_1^\varepsilon) - H(m_2^\varepsilon, s_2^\varepsilon) \geq \phi_1(m_1^\varepsilon) - \phi_2(m_2^\varepsilon)$.

Proof VIII. Estimate of the Hamiltonians

$$
\begin{aligned}
&H(m_1^\varepsilon, s_1^\varepsilon) - H(m_2^\varepsilon, s_2^\varepsilon) \\
&\quad \le \varepsilon^{-1} \langle \hat{\vartheta}[m_1^\varepsilon, m_2^\varepsilon], f(\cdot, m_1^\varepsilon, u_\varepsilon) \rangle_{m_1^\varepsilon} + \varepsilon^{-1} \langle \hat{\vartheta}[m_2^\varepsilon, m_1^\varepsilon], f(\cdot, m_2^\varepsilon, u_\varepsilon) \rangle_{m_2^\varepsilon} \\
&\qquad + \kappa\beta \big[V_g(m_1^\varepsilon) \big]^{\kappa-1} \langle j_g, f(\cdot, m_1^\varepsilon, u_\varepsilon) \rangle_{m_1^\varepsilon} + \kappa\beta \big[V_g(m_2^\varepsilon) \big]^{\kappa-1} \langle j_g, f(\cdot, m_2^\varepsilon, u_\varepsilon) \rangle_{m_2^\varepsilon}.
\end{aligned}
$$

Proof IX. Estimate of the Hamiltonians

Let u_ε be s.t. $H(m_2^\varepsilon, s_2^\varepsilon) = \langle s_2^\varepsilon, f(\cdot, m_2^\varepsilon, u_\varepsilon)\rangle_{m_2^\varepsilon}$.

$$\begin{aligned}
&\varepsilon^{-1}\langle \hat{\vartheta}[m_1^\varepsilon, m_2^\varepsilon], f(\cdot, m_1^\varepsilon, u_\varepsilon)\rangle_{m_1^\varepsilon} + \varepsilon^{-1}\langle \hat{\vartheta}[m_2^\varepsilon, m_1^\varepsilon], f(\cdot, m_2^\varepsilon, u_\varepsilon)\rangle_{m_2^\varepsilon} \\
&\quad = \varepsilon^{-1}\int_{(\mathbb{R}^d)^2} \langle x_1 - x_2, f(x_1, m_1^\varepsilon, u_\varepsilon) - f(x_2, m_2^\varepsilon, u_\varepsilon)\rangle \vartheta[m_1^\varepsilon, m_2^\varepsilon](d(x_1, x_2)) \\
&\quad \leq 2L\varepsilon^{-1} W_2^2(m_1^\varepsilon, m_2^\varepsilon).
\end{aligned}$$

Proof X. Estimate of the Hamiltonians

- Due to [Gurland, 1967, (5)]

$$\begin{aligned}\left|\langle j_g, f(\cdot, m_i^\varepsilon, u)\rangle_{m_i^\varepsilon}\right| &\le 3C_f \int_{\mathbb{R}^d} a(x) g(a(x)) \left[a(x) + \int_{\mathbb{R}^d} a(y) m_i^\varepsilon(dy)\right] m_i^\varepsilon(dx) \\ &\le 6C_f V_g(m_i^\varepsilon).\end{aligned}$$

- Since $\kappa \le 1/(6C_f)$,

$$\beta\left[V_g(m_i^\varepsilon)\right]^{\kappa-1}\kappa\left|\langle j_g, f(\cdot, m_i^\varepsilon, u)\rangle_{m_i^\varepsilon}\right| \le \beta\left[V_g(m_i^\varepsilon)\right]^\kappa.$$

Proof XI. Estimate of the Hamiltonians

- $$H(m_1^\varepsilon, s_1^\varepsilon) - H(m_2^\varepsilon, s_2^\varepsilon) \leq 2L\varepsilon^{-1} W_2^2(m_1^\varepsilon, m_2^\varepsilon) + \beta\big[V_g(m_1^\varepsilon)\big]^\kappa + \beta\big[V_g(m_2^\varepsilon)\big]^\kappa.$$
- Thus,
$$(4L+1)\frac{1}{2\varepsilon} W_2^2(m_1^\varepsilon, m_2^\varepsilon) \geq \Phi_\varepsilon(m_1^\varepsilon, m_2^\varepsilon) \geq \varkappa > 0.$$
- Passing to the limit, we obtain the contadiction.

Kruzhkov transform of the value function

Kruzhkov transform:

$$\phi(m) = 1 - e^{-\mathrm{Val}(m)}.$$

Theorem

The function ϕ is a viscosity solution of Dirichlet problem for Hamilton-Jacobi equation (HJ).

Characterization theorem. Supersolution

- Dynamic programming: for every $\mu \in \operatorname{cl} G$ and $h \in [0, \operatorname{Val}(\mu)]$,

$$\operatorname{Val}(\mu) - h = \inf_{\xi \in \mathcal{U}_h} \operatorname{Val}(m(h; \mu, \xi)).$$

- ϕ is the Krizhkov transform of Val:

$$(1 - \phi(\mu))e^h = \sup_{\xi \in \mathcal{U}_h} (1 - \phi(m(h; \mu, \xi))).$$

- Passing to the limit as $h \downarrow 0$: ϕ is a supersolution of (HJ).

Relaxation of the time optimal problem

Let

$$M^\varepsilon \triangleq \mathrm{cl}\left(\left\{m \in \mathcal{P}^2(\mathbb{R}^d) \text{ s.t. } \mathrm{dist}(m, M) \leq \varepsilon\right\}\right).$$

Given $m(\cdot) \in C([0, +\infty); \mathcal{P}^2(\mathbb{R}^d))$, we set

$$\tau^\varepsilon(m(\cdot)) \triangleq \inf\left\{t \in [0, +\infty) \text{ s.t. } m(t) \in M^\varepsilon\right\}.$$

ε-value:

$$\mathrm{Val}^\varepsilon(\mu) \triangleq \inf\left\{\tau^\varepsilon(m(\cdot; \mu, \xi)) \text{ s.t. } \xi \in \mathcal{U}\right\}.$$

Proprties of the ε-value function

- The function $\mathrm{Val}^{\varepsilon}$ is lower semicontinuous.
- $\mathrm{Val}^{\varepsilon}(\mu) = 0$ whenever $\mu \in M^{\varepsilon}$.
- If $\mu \in G$ is such that $\mathrm{Val}^{\varepsilon}(\mu) < +\infty$. Then, there exists $\xi_{\varepsilon,\mu}$ satisfying

$$\mathrm{Val}^{\varepsilon}(\mu) = \tau^{\varepsilon}(m(\cdot; \mu, \xi_{\varepsilon,\mu})).$$

- (DPP) For every $\mu \in \mathrm{cl}\, G$ and $h \in [0, \mathrm{Val}^{\varepsilon}(\mu)]$,

$$\mathrm{Val}^{\varepsilon}(\mu) - h = \inf_{\xi \in \mathcal{U}_h} \mathrm{Val}(m(h; \mu, \xi)).$$

- If $\varepsilon_1 \leq \varepsilon_2$, then

$$\mathrm{Val}^{\varepsilon_1}(\mu) \geq \mathrm{Val}^{\varepsilon_2}(\mu).$$

Characterization theorem. Subsolution

Let

$$\psi^\varepsilon(\mu) \triangleq 1 - e^{\mathrm{Val}^\varepsilon(\mu)},$$
$$\phi^\varepsilon(\mu) \triangleq \limsup_{\mu' \to \mu,\ \ \mu' \in \mathrm{cl}\, G} \psi^\varepsilon(\mu').$$

DPP for the Val^ε gives that ϕ^ε is a subsolution of (HJ).
Moreover,

$$\phi(\mu) = \lim_{\varepsilon \downarrow 0} \phi^\varepsilon(\mu).$$

Sufficient condition of continuity

Theorem

Assume that the viscosity solution of the Dirichlet problem for HJ PDE is continuous at each point of ∂G*. Then, it is continuous on* $\mathrm{cl}\, G$*.*

Perturbed dynamics

let n be a natural number.

- Dynamics $m_n(\cdot;\mu,\xi)$ obeys

$$\partial_t m(t) + \operatorname{div}\left(\int_U f_n(x, m(t), u) m(t) \xi(du|t)\right) = 0, \quad m(0) = \mu.$$

- Value function

$$\mathrm{Val}_n(\mu) \triangleq \inf\left\{\tau(m_n(\cdot;\mu,\xi) : \xi \in \mathcal{U}\right\}.$$

Γ-convergence

Let X be a topological spae, and let for each natural n, $F_n : X \to \mathbb{R}$.

Definition

The sequence $\{F_n\}_{n=1}^{\infty}$ is said to Γ-converge to $F : X \to \mathbb{R} \cup \{+\infty\}$ if the following two conditions hold:

1. for every sequence $\{x_n\}_{n=1}^{\infty} \in X$ satisfying $\lim_{n\to\infty} x_n = x$,

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n);$$

2. for each $x \in X$, there is a sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } F(x) \geq \limsup_{n\to\infty} F_n(x_n).$$

Limiting value function

Theorem

Let the functions f_n and f be Lipschitz continuous in x and m with a common Lipschitz constant L. In addition, assume the following convergence of the dynamics f_n to f: assume the following convergence of the dynamics f_n to f: for each $c > 0$,

$$\sup \Big\{ \|f_n(x,m,u) - f(x,m,u)\| : \\ x \in \mathbb{R}^d,\ m \in \mathcal{P}^2(\mathbb{R}^d),\ \varsigma(m) \le c,\ u \in U \Big\} \to 0 \text{ as } n \to \infty.$$

Then,

$$\Gamma\text{-}\lim_{n\to\infty} \mathrm{Val}_n = \mathrm{Val}\,.$$

Example

- Dynamics:

$$f(x, m, u) = u - \int_{\mathbb{R}} ym(dy),\ u \in [-1, 0]$$

- Target set:

$$M = \left\{ m \in \mathcal{P}^2(\mathbb{R}) : \int_{\mathbb{R}} ym(dy) = 0 \right\}.$$

In what follows,

$$\bar{m} \triangleq \int_{\mathbb{R}} xm(dx).$$

Example. Hamiltonian

- Hamiltonian:

$$H(m,s) = \inf_{u\in[-1;0]} \int_{\mathbb{R}} s(x)\Big(u-\bar{m}\Big)m(dx)$$
$$= \begin{cases} \Big(-1-\bar{m}\Big)\int_{\mathbb{R}} s(x)m(dx), & \int_{\mathbb{R}} s(x)m(dx) > 0, \\ -\bar{m}\int_{\mathbb{R}} s(x)m(dx), & \int_{\mathbb{R}} s(x)m(dx) \le 0. \end{cases}$$

- Dirichlet problem:

$$H(m,\nabla\phi(m)) + 1 - \phi(m) = 0, \quad \bar{m} \neq 0; \quad \phi(m) = 0 \text{ when } \bar{m} = 0.$$

Example. Solution

- Solution of Dirichlet problem:

$$\phi(\mu) = \begin{cases} 1 - e^{-\ln(1+\bar{\mu})}, & \bar{\mu} \geq 0, \\ 1, & \bar{\mu} < 0. \end{cases}$$

- Value function:

$$\mathrm{Val}(\mu) = \begin{cases} \ln(\bar{\mu} + 1), & \bar{\mu} \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

- Optimal control:
  1. $u(\cdot) \equiv -1$ if $\bar{\mu} \geq 0$;
  2. no optimal control if $\bar{\mu} < 0$;

References I


Ambrosio, L., Gigli, N., and Savaré, G. (2008).
Gradient flows in metric spaces and in the space of probability measures.
Lectures in Mathematics ETH Zürich. Birkhäuser, 2. ed edition.


Badreddine, Z. and Frankowska, H. (2021).
Hamilton–Jacobi inequalities on a metric space.
J. Differ. Equ., 271:1058–1091.


Bardi, M. and Capuzzo-Dolcetta, I. (1997).
Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations.
Birkhäuser Boston, Boston.


Gurland, J. (1967).
The teacher's corner: An inequality satisfied by the expectation of the reciprocal of a random variable.
Am. Stat., 21(2):24–25.

References II



Jimenez, C., Marigonda, A., and Quincampoix, M. (2020).
Optimal control of multiagent systems in the Wasserstein space.
Calc. Var. Partial Differ. Equ., 59(2):45.
Id/No 58.



Subbotin, A. I. (1995).
Generalized solutions of first-order PDEs: The dynamical optimization perspective.
Birkhäuser Boston, Boston.

Thank you for your attention!