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GENERAL ARBITRAGE PRICING MODEL: PROBABILITY AND POSSIBILITY APPROACHES

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Abstract. In this paper, we present a general approach to arbitrage pricing that enables us to treat in a simple way the problems that are typically treated in a rather complicated manner.

The paper has 5 main goals:

1. We present a *general arbitrage pricing model.* It includes as particular cases

- static as well as dynamic models;
- models with a finite number of assets as well as those with an infinite number of assets;
- models related to market risk as well as those related to credit risk;
- models with no transaction costs as well as those with transaction costs;
- models with no costs of short selling as well as those with costs of short selling or with short selling prohibited;
- combinations of various models.

Within the framework of the general arbitrage pricing model, we obtain

- the fundamental theorem of asset pricing;
- the form of fair prices of a contingent claim;
- the form of fair prices of a *controlled contingent claim* (this notion is introduced in the paper).

2. The obtained general results are applied to several particular models (one-period model, multiperiod model, continuous-time model, etc.). The "projection" of general results on these models leads us, in particular, to the revision of the fundamental theorem of asset pricing in the continuous-time case: the proposed variant of this theorem states that the absence of the generalized arbitrage is equivalent to the existence of an equivalent measure, with respect to which the discounted price process is a martingale (not just a sigma-martingale). In the model with the infinite time horizon, uniformly integrable martingales come into play.

3. The general approach mentioned above allows us to narrow considerably the class of equivalent risk-neutral measures (and thus to make the intervals of fair prices shrink) by taking into consideration the current prices of traded securities (options, bonds, etc.).

4. Furthermore, the obtained results are extended to *models with friction*, i.e. models with

- transaction costs;
- restrictions on short selling;
- costs of short selling.

This leads us, in particular to introduction of a notion of *delta-martingale* that generalizes the notions of a martingale, submartingale, and supermartingale.

5. Finally, we introduce the *possibility approach* to arbitrage pricing. When using this approach, one does not need to know the original probability measure. It is shown that all the results described above can be transferred to the possibility framework.

Key words and phrases: American option, arbitrage, arbitrage pricing model, barrier option, Bermudian option, binary option, callable bond, change of numéraire, combination of models, contingent claim, controlled contingent claim, convertible bond, costs of short selling, credit risk, delta-martingale, European option, extendable swap, extracting information from option prices, fair bid-ask price, fair price, filtered possibility space, fundamental theorem of asset pricing, generalized arbitrage, implied volatility, lookback option, lower price, martingale measure, martingale measure with given marginals, measure with given marginals, model with friction, possibility space, puttable bond, puttable swap, risk-neutral measure, set of attainable incomes, set of possible elementary events, sigma-martingale measure, term structure of interest rates, transaction costs, uniformly integrable martingale measure, upper price.

Contents

1 Introduction

1.1 Purpose of the Paper

In the fundamental work [HK79], Harrison and Kreps introduced a general model of pricing by arbitrage. Their paper formed the basis of the martingale approach to arbitrage pricing. However, there are some technical problems inherent in their model. The main one descends from the assumption that the so-called marketed contingent claims should belong to L^2 (the model proposed later by Kreps [K81] enables one to relax this assumption to the L^p-integrability with $p \geq 1$). This restriction is not natural as shown by the example below.

Consider the following simple model for an asset's (discounted) price evolution: $S_0 = 1$, $S_1 = \xi$, $S_2 = \xi\eta$, where ξ and η are independent random variables, each taking on values 1/2 and 3/2 with probability 1/2 (from the financial point of view, S_n is the discounted price of some asset at time n). Let $(\mathcal{F}_n)_{n=0,1,2}$ be a filtration such that $\mathcal{F}_0$ is trivial, S is an $(\mathcal{F}_n)$-martingale, and $\mathcal{F}_1$ is rich enough, so that there exists an $\mathcal{F}_1$-measurable random variable H that is not integrable. Then $H(S_2 - S_1)$ is a natural candidate for a marketed contingent claim. However, it does not belong to L^1.

Further development of arbitrage pricing theory was mainly concentrated on dynamic models with a finite number of assets, which may be viewed as particular cases of the model proposed by Harrison and Kreps. Harrison and Pliska [HP81] introduced the admissibility condition on the trading strategies as a substitute for the integrability restrictions described above. The fundamental theorem of asset pricing (FTAP) for a discrete-time model was established in the papers [HP81] and [DMW90] (alternative proofs were given in [JS98], [KK94], [KS01], [R94], [S92], and [S90]). The FTAP for a continuous-time model was established in the papers [DS94] and [DS98] (another proof was given in [K97]). In a series of papers [DS98], [FK97], [FK98], and [K96], the form of upper and lower prices of a contingent claim in a continuous-time model was established. However, there are some serious problems inherent in the mentioned approach to continuous-time models (these problems are described in Examples 3.6, 3.7, 3.8, and especially Example 3.9).

In this paper, we propose a *general arbitrage pricing model* that has the same spirit as the model of Harrison and Kreps, but avoids the problems described above. This approach allows us to consider in a simple and unified manner various models of the arbitrage pricing theory, some of which have so far been investigated separately and by different techniques (for instance, this concerns the dynamic model with a finite number of assets considered in Section 3.3 and a static model with an infinite number of assets considered in Section 3.6). The simplicity of the proposed approach allows us to investigate in an easy way rather complicated models (like the one described in Section 3.9) as well as models with friction. Our approach to dynamic models with friction described in Section 4.2 is different from an approach developed in a series of papers [DKV02], [KL02], [KRS02], [KS02], and [S04].

The proposed framework enables us not only to treat in a new way the models that have already been studied, but also to go further. In particular, we introduce the *possibility approach* to arbitrage pricing (its essence is described in Section 5.2). This approach avoids the use of such a vague object as the original probability measure P.

1.2 General Arbitrage Pricing Model

A *general arbitrage pricing model* is a quadruple $(\Omega, \mathcal{F}, \mathsf{P}, A)$, where $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space and A (it is called the *set of attainable incomes*) is a collection of random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ meaning the set of discounted incomes one can obtain by trading certain assets. For a model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we introduce the *no generalized arbitrage* (NGA) condition and the notion of an equivalent *risk-neutral measure.* The NGA condition might be viewed as a strengthening of the no free lunch condition known in financial mathematics (the necessity to strengthen the latter one is illustrated by Example 3.26).

The first basic result of the paper is Theorem 2.12, which may be called the fundamental theorem of asset pricing (FTAP) for the general arbitrage pricing model. It states (under some assumption that is automatically satisfied in the particular models considered below) that a model satisfies the NGA condition if and only if there exists an equivalent risk-neutral measure.

We next consider the problem of pricing derivative contracts within the framework of the general arbitrage pricing model. The financial derivatives can be divided into three main categories:

- Contracts whose payoff does not depend on the decisions of either party (holder or issuer). Such are, for example, European options.
- Contracts whose payoff depends on the decisions of one party. Such are, for example, American and Bermudian options, convertible, puttable, and callable bonds, extendable and puttable swaps.
- Contracts whose payoff depends on the decisions of both parties. Such are, for example, some types of swaps.

In this paper, we consider the problem of pricing the contracts of the first type and the second type. We call then *contingent claims* and *controlled contingent claims*, respectively.

A contingent claim is modeled as a random variable F on $(\Omega, \mathcal{F}, \mathsf{P})$ meaning the discounted payoff of the contract. For a contingent claim F, we define the set $I(F)$ of *fair prices*, the set $J(F)$ of *fair bid-ask prices* (this set consists of pairs (x, y), where x stands for the bid price and y stands for the ask price), the *lower price* $V_*(F)$, and the *upper price* $V^*(F)$. The second basic result of the paper is Theorem 2.19. It states (under some natural assumptions) that

$$I(F) \approx \left[\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F,\ \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F\right], \tag{1.1}$$

$$J(F) \approx \left\{(x, y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F\right\}, \tag{1.2}$$

$$V_*(F) = \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_{\mathsf{Q}} F, \tag{1.3}$$

$$V^*(F) = \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_{\mathsf{Q}} F, \tag{1.4}$$

where $\mathcal{R}$ denotes the set of equivalent risk-neutral measures, and the *approximate equality* "$\approx$" between two sets means that these sets coincide up to the border (i.e. their interiors coincide and their closures coincide).

A controlled contingent is modeled as a collection $(F_\lambda)_{\lambda\in\Lambda}$ of random variables on $(\Omega, \mathcal{F}, \mathsf{P})$. The set Λ means the set of controls available to one party and F_λ means the discounted payoff this party obtains if he or she chooses the control λ (this sum might be negative, which means that this party pays the money to the opposite one). For a controlled contingent claim $(F_\lambda)_{\lambda\in\Lambda}$, we define the set $I(F_\lambda; \lambda\in\Lambda)$ of *fair prices*, the set $J(F_\lambda; \lambda\in\Lambda)$ of *fair bid-ask prices*, the *lower price* $V_*(F_\lambda; \lambda\in\Lambda)$, and the *upper price* $V^*(F_\lambda; \lambda\in\Lambda)$. The third basic result of the paper is Theorem 2.25. It states (under some natural assumptions) that

$$I(F_\lambda; \lambda\in\Lambda) \subseteq \Big[\inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}} F_\lambda,\ \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}} F_\lambda\Big], \tag{1.5}$$

$$J(F_\lambda; \lambda\in\Lambda) \approx \Big\{(x,y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}} F_\lambda,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}} F_\lambda\Big\}, \tag{1.6}$$

$$V_*(F_\lambda; \lambda\in\Lambda) = \inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}} F_\lambda, \tag{1.7}$$

$$V^*(F_\lambda; \lambda\in\Lambda) = \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}} F_\lambda. \tag{1.8}$$

The general results described above are presented in Chapter 2, which forms the kernel of the paper. This kernel is further extended in 3 directions:

1. "Projecting" general results on particular models with no transaction costs, with short selling, and with no costs of short selling (Chapter 3).
2. Extending the results of Chapter 3 to models with transaction costs, restrictions on short selling, or costs of short selling (Chapter 4).
3. Introducing the possibility approach to asset pricing and transferring the obtained results to the possibility framework (Chapter 5).

1.3 Particular Models with No Friction

Various models of arbitrage pricing can be viewed as particular cases of the general model described above. In order to embed a particular model into this general framework, one should

1. specify the set A of attainable incomes;
2. find out the structure of the set of equivalent risk-neutral measures (typically, the risk-neutral measures in a particular model admit a simpler description than the general definition of a risk-neutral measure).

Once this is done, Theorem 2.12 gives the necessary and sufficient conditions for the absence of the generalized arbitrage, while formulas (1.1)–(1.8) yield the form of the sets of fair prices of contingent claims and controlled contingent claims.

In Chapter 3, we consider 9 particular *models with no friction*, i.e. the models with

- no transaction costs;
- no restrictions on short selling;
- no costs of short selling.

The word "particular" reflects the fact that they can be viewed as particular cases of the general model introduced in Chapter 2, but these models are general in the sense that neither of them imposes restrictions on the probabilistic structure of the asset price evolution (like the assumption that the price process is a geometric Brownian motion, etc.).

In Section 3.1, we consider the one-period model with a finite number of assets and show that for this model the NGA condition is equivalent to the classical no arbitrage (NA) condition, while the class of equivalent risk-neutral measures coincides with the class of martingale measures. Furthermore, our interval of fair prices $I(F)$ coincides with the classical interval of fair prices of a contingent claim F. In other words, the "projection" of the results of Chapter 2 on this model agrees with the classical results (they are described in Section 2.1).

In Section 3.2, similar results are obtained for the multiperiod model.

Section 3.3 deals with the continuous-time model with a finite time horizon. Our approach to this model turns out to be completely different from the traditional approach. First, we consider only simple (i.e. piecewise constant) trading strategies with no admissibility condition imposed. Second, our FTAP states that the model satisfies the NGA condition if and only if there exists an equivalent *martingale measure*, i.e. a measure, with respect to which the discounted price process is a martingale. This is completely different from the traditional FTAP provided by Delbaen and Schachermayer [DS94], [DS98] (another proof was given by Kabanov [K97]), which states that a model satisfies the no free lunch with vanishing risk (NFLVR) condition (defined through the general predictable admissible strategies) if and only if there exists an equivalent *sigma-martingale measure*, i.e. a measure, with respect to which the discounted price process is a sigma-martingale (this class of processes has been introduced by Chou [C79] and Émery [E80]). Third, our definition of the interval of fair prices differs from the traditional one. We discuss in Section 3.3 the problems of the traditional theory of arbitrage pricing that arise when one considers admissible strategies, sigma-martingale (not martingale) measures, and traditional intervals of fair prices. These problems do not arise in our framework. The proposed FTAP agrees with the result of Yan [Y98] who proved that some sort of no-arbitrage condition (introduced in [Y98]) is equivalent to the existence of a martingale measure. It is also shown that the proposed approach is consistent with the Black–Scholes formula as well as with its extensions to dividend-paying stocks (Merton's formula) and to futures (Black's formula). Furthermore, it turns out that a change of numéraire preserves the NGA property (see Theorem 3.11). In contrast, the NFLVR property is not preserved under the change of numéraire (see Example 3.10, which is borrowed from [DS95]).

Section 3.4 is related to the continuous-time model with the infinite time horizon. It is proved that the set $\mathcal{R}$ coincides with the set of equivalent *uniformly integrable martingale measures*, i.e. the measures, with respect to which the discounted price process is a uniformly integrable martingale. Again, our approach is different from the traditional one. Our FTAP provides, in particular, an explanation of the paradox of the "buy and hold" strategies. As an application of the obtained results, we show that any model with the infinite time horizon, in which the logarithmic price process has stationary increments, does not satisfy the NGA condition (even if the price process is already a martingale with respect to the original measure).

Section 3.5 deals with the model of the term structure of interest rates. A distinctive feature of this model is that it has an infinite number of assets. As we are dealing only with the simple strategies, no problems arise in defining the set of attainable incomes (in the traditional approach, a serious problem is how to define the stochastic integral of a general predictable strategy with respect to an infinite-dimensional process; see [BDKR97]). We prove, in particular, that this model satisfies the NGA condition if and only if there exists an equivalent measure, with respect to which the discounted price of each risk-free zero-coupon bond is a martingale. This property is typically taken as the definition of no arbitrage in such a model (see [MR97; Def. 12.1.1]).

The models of Sections 3.6–3.9 are unified by the same methodology. Let us describe it. Any arbitrage pricing model used in modern financial mathematics works as follows: its output is the set of fair prices of a contract that is being introduced into the market; its first input is the measure P that describes the probability structure of the price evolution; its second input consists of the current market prices of the traded contracts, including both primary financial instruments (shares, bonds) and secondary ones (options, swaps, etc.).

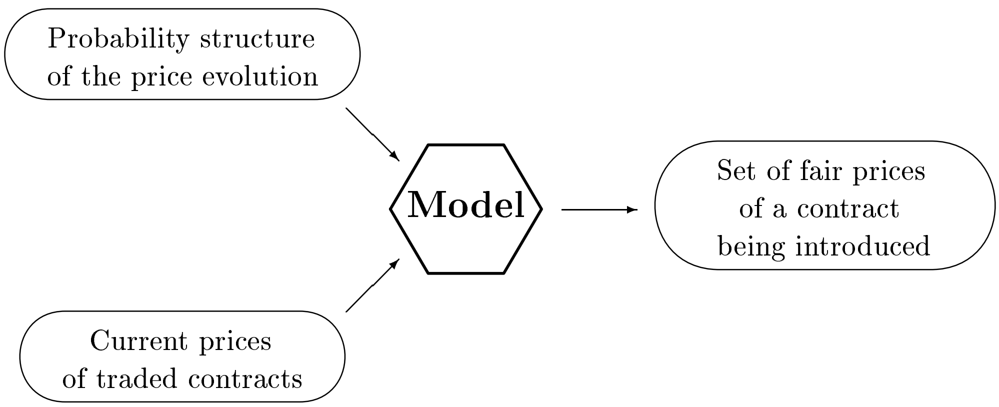

Figure 1. Inputs and output of an arbitrage pricing model

Typically, the second input is used to evaluate the risk-neutral measure. This can be done in several ways. In one method, the risk-neutral measures are assumed to depend on several parameters, and the current prices of traded contracts are used to evaluate these parameters (for example, the Black–Scholes model is applied in this way). Another method is to evaluate the risk-neutral measure non-parametrically. This technique is typically applied to recover the risk-neutral distribution of an asset's price at time T from the current prices of European options on this asset with maturity T. The corresponding model was first considered by Breeden and Litzenberger [BL78]. An overview of the existing literature on this topic is given in [J99].

In the present paper, the second input is used to restrict the class of equivalent risk-neutral measures. This is done within the framework of the general arbitrage pricing model as follows. The set A depends on the amount of traded securities that we take into account; the set $\mathcal{R}$ depends on A; the sets of fair prices depend on $\mathcal{R}$ through (1.1)–(1.8). Diagramatically,

$$\text{Assets} \longrightarrow A \longrightarrow \mathcal{R} \longrightarrow \text{Sets of fair prices.}$$

When the amount of assets taken into consideration is enlarged (i.e. more prices of traded contracts are taken into account), the set A is enlarged, the set $\mathcal{R}$ is reduced, and the sets of fair prices are reduced. In Sections 3.6–3.9, we show what information on the structure of $\mathcal{R}$ can be extracted from the prices of traded options and bonds.

Section 3.6 deals with the model that takes into account the current prices of traded European call options on several assets with maturity T. We show that if options with all positive strike prices are traded (of course, this is an idealized assumption, but it is often used in theory), then the set $\mathcal{R}$ consists of the equivalent *measures with given marginals*, i.e. the measures, with respect to which the vector $(S_T^1, \dots, S_T^d)$ of prices of these assets at time T has preassigned marginal distributions (these distributions are extracted from the option prices). To put it another way, by looking at the prices of the European call options on some asset with a fixed maturity T and all positive strike prices, one can recover the market-estimated distribution of the asset price at time T. As a corollary, the fair price of a contingent claim of the form $F = f(S_T^i)$ (such are, for example, the binary options) is uniquely determined through the NGA considerations by the prices of the European call options on the i-th asset with maturity T and all positive strike prices. This extends the result of Breeden and Litzenberger [BL78].

Section 3.7 is related to the model, which takes into account the current prices of traded barrier options. It is shown that by looking at the current prices of the up-and-in call options on some asset with maturity T, all positive barriers and all positive strike prices, one can recover the market-estimated distribution of the pair (S_T, M_T). Here S_T is the price of the underlying asset at time T, and M_T is the maximal price of this asset on the interval $[0, T]$. As a corollary, the fair price of a contingent claim of the form $F = f(S_T, M_T)$ (such are, for example, the lookback options) is uniquely determined through the NGA considerations by the prices of the up-and-in call options on this asset with maturity T, all positive barriers, and all positive strike prices.

Section 3.8 deals with a model for assessing credit risk. We show (under some assumptions that might be disputed) that the market-estimated distribution of the default time of a company is determined by the current prices of (risky) zero-coupon bonds with all positive maturities issued by this company.

The general approach introduced in Chapter 2 admits an easy procedure of the *combination of models*. The aim of this procedure is to narrow the sets of fair prices by taking into consideration the current prices of a larger amount of traded contracts. Thus, the models of Section 3.1–3.8 may be viewed as "building blocks" for constructing mixed models. An example is provided by Section 3.9, in which we consider a mixed static-dynamic model. The "building blocks" are provided by the models of Sections 3.3 and 3.6. We show that, for the mixed model, the set $\mathcal{R}$ consists of the equivalent *martingale measures with given marginals*, i.e. the measures, with respect to which the discounted price process is a martingale with preassigned marginal distributions. Such measures have recently attracted attention in the literature (see [C04], [CGMY03; Sect. 4.1], and [MY02]).

1.4 Particular Models with Friction

The general approach introduced in Chapter 2 can be applied not only to the idealized models of Chapter 3, but also to more realistic models with

- transaction costs;
- restrictions on short selling;
- costs of short selling.

In Chapter 4, we consider the "duals" of the models of Chapter 3 that take these 3 effects into account. All 3 effects are included in the models simultaneously. We call the corresponding extensions *models with friction*. In order to embed such a model into the general framework of Chapter 2, one should

1. specify the set A of attainable incomes;
2. find out the structure of equivalent risk-neutral measures.

Once this is done, Theorem 2.12 gives the necessary and sufficient conditions for the absence of the generalized arbitrage, while formulas (1.1)–(1.8) yield the form of the sets of fair prices of contingent claims and controlled contingent claims.

In Section 4.2, we consider a continuous-time finite-horizon model with friction. This is done on three levels of generality. First, we find the structure of risk-neutral measures for a model with arbitrary transaction costs and arbitrary costs of short-selling. Then we consider a model with arbitrary transaction costs and no costs of short selling and prove that in this model a measure $\mathsf{Q} \sim \mathsf{P}$ is a risk-neutral measure if and only if there exists a Q-martingale that is (componentwise) between the discounted ask and bid price processes. Finally, we consider a model with proportional transaction costs and proportional costs of short selling. We introduce the notion of a *delta-martingale* (this generalizes the notions of a martingale, supermartingale, and submartingale) and prove that for this model the class of equivalent risk-neutral measures coincides with the class of equivalent *delta-martingale measures*, i.e. the measures, with respect to which the discounted ask (equivalently, bid) price process is a delta-martingale.

Our approach to arbitrage pricing in the models with transaction costs appears to be different from other approaches known in the literature (see, for example, [CK96], [CPT99], [DKV02], [JK95], [KL02], [KRS02], [KS02], and [S04]). One of the important differences between our approach and the one developed in the papers [CK96], [CPT99], [DKV02], [KL02], and [KS02] is as follows. In these papers, a contingent claim is defined as a d-dimensional random vector (its i-th component means the amount of assets of the i-th type received by the holder of the contingent claim). The substitute for the upper (resp., lower) price is the region in $\mathbb{R}^d$ of the initial endowments (these are random vectors whose i-th component means the amount of assets of the i-th type in the initial portfolio), from which the contingent claim can be superreplicated (resp., subreplicated). In contrast, a contingent claim here is a real-valued random variable, and we deal with its lower and upper prices, which are real numbers as in the frictionless models. In this respect, our approach to models with friction is similar to that in [JK95], but there is a number of other essential differences between the two approaches (see remark following Key Lemma 4.8). In particular, the above mentioned papers take only the transaction costs into account, and there is a number of other papers related to restrictions on short sales, while in our approach these effects as well as the costs of short selling are taken into account simultaneously.

1.5 Possibility Approach

When a coin is tossed, everyone agrees that there exists a probability measure on the set of elementary outcomes, and this measure assigns the mass 1/2 to each of the two outcomes. When shooting at a target is performed, everyone agrees that there exists a probability measure on the set of elementary outcomes. The exact form of this measure cannot be found by pure thought, but can be estimated by repeating the trials. In both examples, the legitimacy of a probability measure is based on the existence of a fixed set of conditions that admits an unlimited number of repetitions. The importance of such a set of conditions has been stressed by Kolmogorov [K33; Ch. I, § 2].

In the problems that finance deals with, such a fixed set of conditions does not seem to exist at all. Therefore, it is questionable whether there exists a measure P, which serves as the first input to an arbitrage pricing model (see Figure 1). It is unquestionable that even if such a measure exists, then no one knows exactly what it is. In other words, the first input to an arbitrage pricing model displayed in Figure 1 is rather vague. In contrast, the second input is absolutely solid since the current prices of traded contracts are observed directly.

We introduce the *possibility approach* to arbitrage pricing. It requires as the first input the set of all possible outcomes and does not require the probabilities assigned to these outcomes. To be more precise, the possibility approach is based on the *possibility space* $(\Omega, \mathcal{F})$ instead of the probability space $(\Omega, \mathcal{F}, \mathsf{P})$. We call Ω the *set of possible elementary events.* Usually it can be defined by pure thought (i.e. without using the real data) in an unambiguous way. For example, a natural set of possible prices of some asset is $\mathbb{R}_{++} (= (0, \infty))$; a natural set of possible prices of d assets is $\mathbb{R}^d_{++}$. Typically, the set of possible elementary events admits a natural topology, and $\mathcal{F}$ is taken as its Borel σ-field.

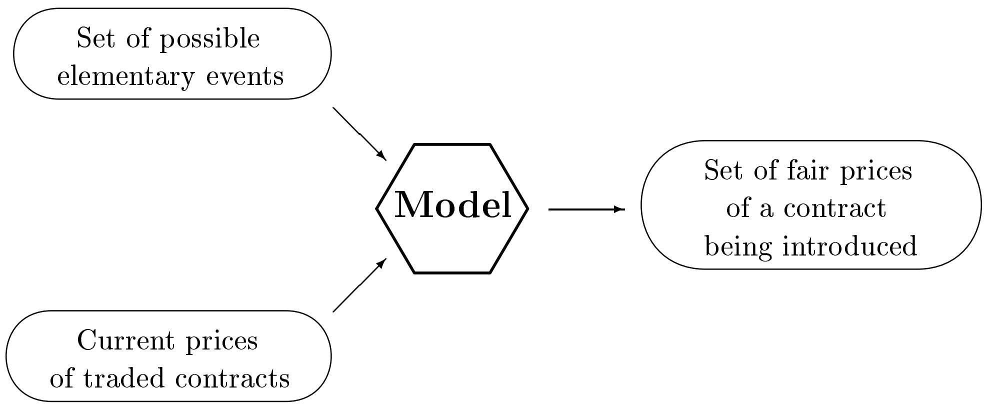

Figure 2. Inputs and output of a possibility arbitrage pricing model

Let us compare the possibility approach to pricing by arbitrage with other pricing techniques, namely:

- equilibrium pricing,
- optimality pricing,
- arbitrage pricing in the probability setting.

(For more information on these techniques, see, for example [D01].) Equilibrium pricing requires as the inputs the utility function and the initial capital of each market participant, the probability structure of the price evolution, and the current prices of traded contracts. This technique yields as the output the equilibrium price of a new contract. Optimality pricing requires as the inputs the utility function and the initial capital of some market participant, the probability structure of the price evolution, and the current prices of traded contracts. This technique yields as the output the fair price of a contract for this market participant. Arbitrage pricing in the probability setting requires as the inputs the probability structure of the price evolution and the current prices of traded contracts. This technique yields as the output the interval of fair prices for a new contract. Arbitrage pricing in the possibility setting requires as the input the current prices of traded contracts. It yields as the output the interval of fair prices for a new contract, which is wider than the "probability" interval, but is "safer" (see Figure 3). Thus, arbitrage pricing in the possibility setting appears to be the most "robust" of pricing techniques.

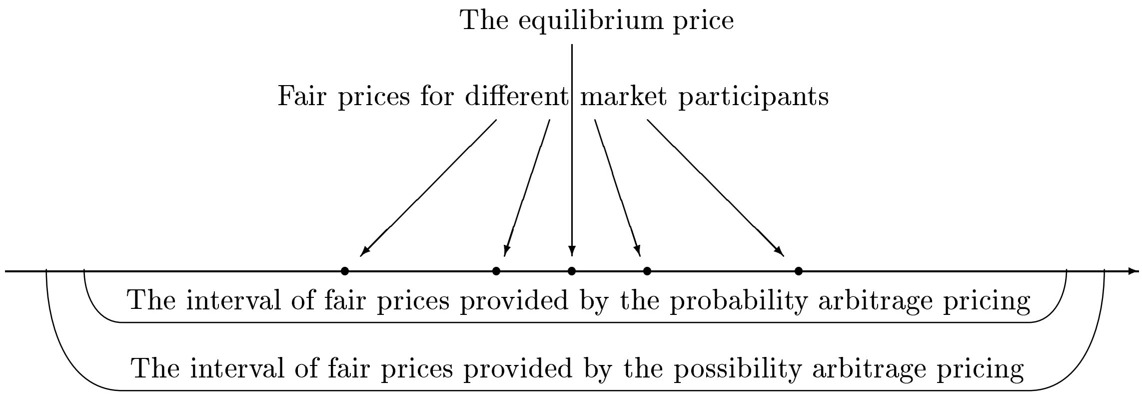

Figure 3. Comparison of various pricing techniques

Let us remark that some examples of arbitrage pricing with no use of probability measure can be found in financial mathematics and financial practice. One example is the calculation of exchange rates through the triangular arbitrage. Another example is the calculation of prices of contingent claims in the one-period model (see Examples 5.1–5.4). More complicated models have been considered in [BL78], [BHR01], [H98].

The possibility approach is introduced in Chapter 5. It is shown that all the notions and the main results of Chapters 2, 3, and 4 can be transferred to the possibility framework.

A general arbitrage pricing model in the possibility setting is a triple $(\Omega, \mathcal{F}, A)$, where $(\Omega, \mathcal{F})$ is a possibility space and A is a collection of real-valued $\mathcal{F}$-measurable functions. The interpretation of A is the same as in the probability framework. We define the possibility version of the NGA condition and of a risk-neutral measure.

Theorem 5.17 states (under some assumption) that a model satisfies the NGA condition if and only if for any $D \in \mathcal{F} \setminus \{\emptyset\}$, there exists a risk-neutral measure Q such that $\mathsf{Q}(D) > 0$. Thus, in "fair" models the risk-neutral measure should exist even if we do not assume the existence of the original measure P. We believe that this result agrees with practice, where the risk-neutral measure arises as the weighted average of the market participants' expectations deformed in accordance with their risk aversion (for more details, see [RJ04]). A nice illustration is provided by the bookmaking, where the "true" distribution on the set of outcomes is completely unclear, while the "market-estimated" distribution is easily recovered from the bets.

The objects $I(F)$, $J(F)$, $V_*(F)$, and $V^*(F)$ are appropriately redefined in the possibility framework. Theorem 5.23 states (under some natural assumptions) that equalities (1.1)–(1.4) remain valid (now, $\mathcal{R}$ denotes the set of risk-neutral measures with the word "equivalent" dropped).

Theorem 5.28 contains a similar result for equalities (1.5)–(1.8).

To sum up, when applying the possibility approach to arbitrage pricing, one should

1. specify the possibility space $(\Omega, \mathcal{F})$;
2. specify the set A of attainable incomes;
3. find out the structure of the set of risk-neutral measures.

Once this is done, Theorem 5.17 gives the necessary and sufficient conditions for the absence of the generalized arbitrage, while formulas (1.1)–(1.8) yield the form of the sets of fair prices of contingent claims and controlled contingent claims.

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Model	Probability approach		Possibility approach	
	Model with no friction	**Model with friction**	**Model with no friction**	**Model with friction**
General arbitrage pricing model	2.2, 2.3, 2.4		5.3, 5.4, 5.5	
One-period model	3.1	4.1	+	+
Multiperiod model	3.2	+	5.7	5.7
Continuous-time model with a finite time horizon	3.3	4.2	5.8	5.8
Continuous-time model with the infinite time horizon	3.4	−	5.9	−
Model of the term structure of interest rates	3.5	+	+	+
Model with European options	3.6	4.3	5.10	5.10
Model with barrier options	3.7	+	+	+
Model for assessing credit risk	3.8	+	+	+
Mixed model	3.9	4.4	+	+

Table 1. The structure table for the models considered in the paper. The numbers indicate the sections, in which the corresponding model is investigated (for example, the general arbitrage pricing model in the possibility setting is investigated in Sections 5.3, 5.4, and 5.5). The sign "+" means that the corresponding model is not considered explicitly in the paper, but it can be analyzed by analogy with some other model considered in the paper. The sign "−" means that the investigation of the corresponding model is still an open question.

<table>
<tr><th>Results</th><th>Probability approach</th><th>Possibility approach</th></tr>
<tr><td>FTAP</td><td>Under Assumption 2.11,
NGA $\iff$ $\mathcal{R} \neq \emptyset$.</td><td>Under Assumption 5.16,
NGA $\iff$
$\forall D \in \mathcal{F} \setminus \{\emptyset\}, \exists \mathsf{Q} \in \mathcal{R} : \mathsf{Q}(D) > 0$.</td></tr>
<tr><td rowspan="2">Pricing of contingent claims</td><td>If Assumption 2.11 and the NGA condition are satisfied, while F is bounded below, then</td><td>If Assumption 5.16 and the NGA condition are satisfied, while F is bounded below and $\mathsf{E_Q} F < \infty$ for any $\mathsf{Q} \in \mathcal{R}$, then</td></tr>
<tr><td colspan="2">$$I(F) \approx \left[\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q} F,\ \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q} F\right],$$ $$J(F) \approx \left\{(x,y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q} F,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q} F\right\},$$ $$V_*(F) = \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q} F,$$ $$V^*(F) = \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q} F.$$</td></tr>
<tr><td rowspan="2">Pricing of controlled contingent claims</td><td>If Assumption 2.11 and the NGA condition are satisfied, while F_λ is bounded below for any $\lambda \in \Lambda$, then</td><td>If Assumption 5.16 and the NGA condition are satisfied, while F_λ is bounded below for any $\lambda \in \Lambda$ and $\sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda < \infty$ for any $\mathsf{Q} \in \mathcal{R}$, then</td></tr>
<tr><td colspan="2">$$I(F_\lambda; \lambda \in \Lambda) \subseteq \left[\inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda,\ \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda\right],$$ $$J(F_\lambda; \lambda \in \Lambda) \approx \left\{(x,y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda\right\},$$ $$V_*(F_\lambda; \lambda \in \Lambda) = \inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda,$$ $$V^*(F_\lambda; \lambda \in \Lambda) = \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda.$$</td></tr>
</table>

Table 2. The results obtained for the general arbitrage pricing model in the probability setting and in the possibility setting. The table shows that for each model there are 9 main results: FTAP, the form of $I(F)$, $J(F)$, $V_*(F)$, $V^*(F)$, $I(F_\lambda; \lambda \in \Lambda)$, $J(F_\lambda; \lambda \in \Lambda)$, $V_*(F_\lambda; \lambda \in \Lambda)$, and $V^*(F_\lambda; \lambda \in \Lambda)$. These results are "projected" on particular models described in Table 1. As there are 36 "solved" models (including the general ones), there are $324 = 9 \times 36$ "pricing results" presented in the paper (in fact, a few more).

2 General Arbitrage Pricing Model

Section 2.1 may be viewed as a preliminary step before introducing the general arbitrage pricing model. In this section, we describe the classical approach to arbitrage pricing in a one-period model with a finite number of assets. This material is well known (for more details, one may consult, for instance, [FS02; Ch. 1]).

The general arbitrage pricing model introduced in Section 2.2 may be regarded as the infinite-dimensional version of the model of Section 2.1 (with the definitions of no arbitrage and the definitions of fair prices appropriately reformulated).

2.1 Ordinary Arbitrage in a One-Period Model

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Let $S_0 \in \mathbb{R}^d$ and S_1 be an $\mathbb{R}^d$-valued random vector on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, S_n^i is the price of the i-th asset at time n. This asset might be

- a traded asset (bond, stock, option, commodity, etc.) providing no dividends;
- a dividend-paying stock, stock index, or a foreign currency;
- a futures (in this case S_n^i is the futures price)

(for financial details, see [H97]). Let $r \in \mathbb{R}_+$ be the risk-free interest rate and $q^i \in \mathbb{R}_+$ be the dividend rate on the i-th asset in the case, where this asset is a dividend-paying stock (the dividend paid at time 1 is $q^i S_1^i$), stock index (q^i is then the weighted average of dividend rates of the stocks in the index), or a foreign currency (q^i is then the risk-free interest rate on this currency). Define the *discounted price* of the i-th asset by

$$\overline{S}_n^i = \begin{cases} \dfrac{S_n^i}{(1+r)^n} & \text{if the } i\text{-th asset is a traded asset providing no dividends,} \\ \dfrac{(1+q^i)^n}{(1+r)^n} S_n^i & \text{if the } i\text{-th asset is a dividend-paying stock, stock index, or a foreign currency,} \\ S_n^i & \text{if the } i\text{-th asset is a futures.} \end{cases} \tag{2.1}$$

Consider the set

$$A = \left\{ \sum_{i=1}^d h^i \big(\overline{S}_1^i - \overline{S}_0^i\big) : h^i \in \mathbb{R} \right\}. \tag{2.2}$$

From the financial point of view, A is the set of incomes discounted to time 0 that can be obtained by trading assets $1, \dots, d$ at times $0,\ 1$ (and using the bank account to borrow/lend money). Indeed, if the i-th asset is a traded asset providing no dividends, then in order to buy one asset at time 0, one should pay the amount $S_0^i = \overline{S}_0^i$, while at time 1 the sum S_1^i is obtained; the latter amount discounted to time 0 is $\overline{S}_1^i$, so the discounted income is $\overline{S}_1^i - \overline{S}_0^i$. If the i-th asset provides dividends, then its owner obtains at time 1 the amount $(1+q^i)S_1^i$, so the discounted income is again $\overline{S}_1^i - \overline{S}_0^i$. If the i-th asset is a futures, then a person who takes a long position pays nothing at time 0 and obtains the amount $S_1^i - S_0^i = \overline{S}_1^i - \overline{S}_0^i$ at time 1, so the discounted income is $(1+r)^{-1}\big(\overline{S}_1^i - \overline{S}_0^i\big)$. Similar reasoning can be applied to the incomes obtained by the short selling of the asset or by taking a short futures position. (We consider here the frictionless model.)

Definition 2.1. A *one-period model* is a collection $(\Omega, \mathcal{F}, \mathsf{P}, \overline{S}_0, \overline{S}_1)$.

Definition 2.2. A model $(\Omega, \mathcal{F}, \mathsf{P}, \overline{S}_0, \overline{S}_1)$ satisfies the *no arbitrage* (NA) condition if $A \cap L_+^0 = \{0\}$ (L_+^0 denotes the set of $\mathbb{R}_+$-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$).

`Remark`. The random variables are considered as the classes of equivalence under the indistinguishability relation.

Definition 2.3. An equivalent *martingale measure* is a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q}|\overline{S}_1| < \infty$ and $\mathsf{E}_\mathsf{Q}\overline{S}_1 = \overline{S}_0$. The set of equivalent martingale measures will be denoted by $\mathcal{M}$.

`Notation`. Set $C = \overline{\operatorname{conv}} \operatorname{supp} \operatorname{Law}_\mathsf{P} \overline{S}_1$, where "$\overline{\operatorname{conv}}$" denotes the closed convex hull, "supp" denotes the support, and $\operatorname{Law}_\mathsf{P} \overline{S}_1$ is the distribution of $\overline{S}_1$ under P. Let C° denote the *relative interior* of C, i.e. the interior of C in the relative topology of the smallest affine subspace of $\mathbb{R}^d$ containing C.

Theorem 2.4 (FTAP). *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, \overline{S}_0, \overline{S}_1)$, *the following conditions are equivalent:*

(a) *NA*;
(b) $\overline{S}_0 \in C^\circ$;
(c) $\mathcal{M} \neq \emptyset$.

`Proof`. *Step 1.* Let us prove the implication (a)$\Rightarrow$(b). If $\overline{S}_0 \notin C^\circ$, then, by the separation theorem, there exists a vector $h \in \mathbb{R}^d$ such that $\langle h, (x - \overline{S}_0)\rangle \geq 0$ for any $x \in C$ and $\langle h, (x - \overline{S}_0)\rangle > 0$ for some $x \in C$. This means that $\langle h, (\overline{S}_1 - \overline{S}_0)\rangle \geq 0$ P-a.s. and $\mathsf{P}(\langle h, (\overline{S}_1 - \overline{S}_0)\rangle > 0) > 0$. But this contradicts the NA condition.

Step 2. Let us prove the implication (b)$\Rightarrow$(c). The set

$$E = \{\mathsf{E}_\mathsf{Q}\overline{S}_1 : \mathsf{Q} \sim \mathsf{P},\ \mathsf{E}_\mathsf{Q}|\overline{S}_1| < \infty\}$$

is convex, and the closure of E contains $\operatorname{supp} \operatorname{Law}_\mathsf{P} \overline{S}_1$. Consequently, $E \supseteq C^\circ$.

Step 3. Let us prove the implication (c)$\Rightarrow$(a). Take $\mathsf{Q} \in \mathcal{M}$. Then $\mathsf{E}_\mathsf{Q}X = 0$ for any $X \in A$. This implies the NA condition. □

Now, let F be a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, F is the payoff of some contingent claim discounted to time 0.

Definition 2.5. (i) A real number x is a *fair price* of F if the model with $d+1$ assets $(\Omega, \mathcal{F}, \mathsf{P}, x, \overline{S}_0^1, \dots, \overline{S}_0^d, F, \overline{S}_1^1, \dots, \overline{S}_1^d)$ satisfies the NA condition. The set of fair prices of F will be denoted by $I(F)$.

(ii) The *lower* and *upper* prices of F are defined by

$$V_*(F) = \inf\{x : x \in I(F)\},$$
$$V^*(F) = \sup\{x : x \in I(F)\}.$$

`Notation`. Set $D = \overline{\operatorname{conv}} \operatorname{supp} \operatorname{Law}_{\mathsf{P}}(F, \overline{S}_1)$ and let D° denote the relative interior of D.

Theorem 2.6 (Main theorem for pricing contingent claims). *Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, \overline{S}_0, \overline{S}_1)$ satisfies the NA condition. Then*

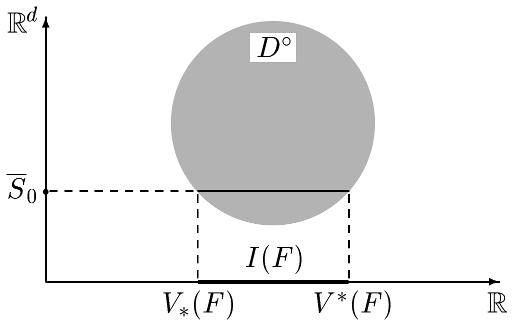
$$I(F) = \{x : (x, \overline{S}_0) \in D^\circ\} = \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{M}\}, \tag{2.3}$$

$$V_*(F) = \inf_{\mathsf{Q} \in \mathcal{M}} \mathsf{E}_{\mathsf{Q}} F, \tag{2.4}$$

$$V^*(F) = \sup_{\mathsf{Q} \in \mathcal{M}} \mathsf{E}_{\mathsf{Q}} F. \tag{2.5}$$

The expectation $\mathsf{E}_{\mathsf{Q}} F$ here is taken in the sense of finite expectations, i.e. we consider only those Q, for which $\mathsf{E}_{\mathsf{Q}}|F| < \infty$.

`Proof`. Equality (2.3) is a straightforward consequence of Theorem 2.4. Equalities (2.4) and (2.5) follow from (2.3). □



`Figure 4.` The joint arrangement of $I(F)$, $V_*(F)$, $V^*(F)$, and D°

`Remark`. Another way to define the lower and upper prices (which is commonly used in financial mathematics) is as follows:

$$C_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \le F \ \mathsf{P}\text{-a.s.}\},$$
$$C^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \ge F \ \mathsf{P}\text{-a.s.}\}.$$

One can easily check that if the model $(\Omega, \mathcal{F}, \mathsf{P}, \overline{S}_0, \overline{S}_1)$ satisfies the NA condition, then $C_*(F) = V_*(F)$ and $C^*(F) = V^*(F)$ (a proof can be found in [FS02; Th. 1.23]).

2.2 Generalized Arbitrage

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space.

Definition 2.7. An *arbitrage pricing model* is a quadruple $(\Omega, \mathcal{F}, \mathsf{P}, A)$, where A is a convex cone in L^0 (L^0 is the space of real-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ considered up to indistinguishability). The set A will be called the *set of attainable incomes.*

From the financial point of view, A is the set of all incomes discounted to the initial time that can be obtained by trading a certain amount of assets (and using the bank account to borrow/lend money). An example is provided by (2.2). In the frictionless models, A is a linear space. In the models with friction, A is a cone.

`Notation`. (i) Set

$$B = \Big\{ Z \in L^0 : \text{ there exist } (X_n)_{n\in\mathbb{N}} \in A \text{ and } a \in \mathbb{R} \\ \text{such that } X_n \geq a \text{ P-a.s. and } Z = \lim_{n\to\infty} X_n \text{ P-a.s.} \Big\}. \tag{2.6}$$

The elements of B might be regarded as generalized attainable incomes bounded below.

(ii) For $Z \in B$, denote $\gamma(Z) = 1 - \operatorname{essinf}_{\omega\in\Omega} Z(\omega)$ and set

$$\begin{aligned} A_1 &= \{X - Y : X \in A,\ Y \in L^0_+\}, \\ A_2(Z) &= \left\{ \frac{X}{Z + \gamma(Z)} : X \in A_1 \right\}, \\ A_3(Z) &= A_2(Z) \cap L^\infty, \\ A_4(Z) &= \text{closure of } A_3(Z) \text{ in } \sigma(L^\infty, L^1(\mathsf{P})). \end{aligned} \tag{2.7}$$

Here L^0_+ is the set of $\mathbb{R}_+$-valued elements of L^0; L^∞ is the space of bounded elements of L^0; $\sigma(L^\infty, L^1(\mathsf{P}))$ denotes the weak topology on L^∞ induced by the space $L^1(\mathsf{P})$ of the P-integrable random variables on $(\Omega, \mathcal{F}, \mathsf{P})$.

Definition 2.8. A model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the *no generalized arbitrage* (NGA) condition if for any $Z \in B$, we have $A_4(Z) \cap L^0_+ = \{0\}$.

`Remarks`. (i) Note that $A_4(Z) \cap L^0_+ = \{0\}$ if and only if $A_5(Z) \cap L^0_+ = \{0\}$, where

$$A_5(Z) = \{(Z + \gamma(Z))X : X \in A_4(Z)\}. \tag{2.8}$$

The elements of $A_5(Z)$ might be regarded as generalized attainable incomes (i.e. one can approximate the elements of $A_5(Z)$ by the elements of A_1).

(ii) The existence of a generalized arbitrage opportunity means that there exist $Z \in B$, $W \in L^0_+ \setminus \{0\}$ and generalized sequences $(X_\lambda)_{\lambda\in\Lambda} \in A$, $(Y_\lambda)_{\lambda\in\Lambda} \in L^0_+$, and $(\alpha_\lambda)_{\lambda\in\Lambda} \in \mathbb{R}_+$ such that $|X_\lambda - Y_\lambda| \leq \alpha_\lambda(Z + \gamma(Z))$, $\lambda \in \Lambda$ and $(X_\lambda - Y_\lambda)$ converges to W in the sense that $\mathsf{E}_\mathsf{Q}(X_\lambda - Y_\lambda) \longrightarrow \mathsf{E}_\mathsf{Q} W$ for any probability measure $\mathsf{Q} \ll \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} Z < \infty$.

(iii) The NGA condition is similar to the *no free lunch* (NFL) condition introduced by Kreps [K81] in a different framework. The NFL condition can be defined in our framework as: $A_4(0) \cap L^0_+ = \{0\}$. One can also define the *no arbitrage* (NA) condition in our framework as: $A \cap L^0_+ = \{0\}$. The NGA condition is the strongest one: NGA$\Rightarrow$NFL, NGA$\Rightarrow$NA.

Definition 2.9. An equivalent *risk-neutral measure* is a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X^- \geq \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$ (we use the notation $X^- = -X \vee 0$, $X^+ = X \vee 0$). The expectations $\mathsf{E}_\mathsf{Q} X^-$ and $\mathsf{E}_\mathsf{Q} X^+$ here may take on the value $+\infty$. The set of equivalent risk-neutral measures will be denoted by $\mathcal{R}$.

`Notation`. For $Z \in B$, we will denote by $\mathcal{R}(Z)$ the set of the probability measures $\mathsf{Q} \sim \mathsf{P}$ with the property: for any $X \in A$ such that $X \geq -\alpha Z - \beta$ P-a.s. with some $\alpha, \beta \in \mathbb{R}_+$, we have $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \leq 0$.

Lemma 2.10. *For any $Z \in B$, we have $\mathcal{R} \subseteq \mathcal{R}(Z)$.*

`Proof.` Take $\mathsf{Q} \in \mathcal{R}$. It follows from the Fatou lemma that Z is Q-integrable. Thus, if $X \in A$ satisfies the inequality $X \geq -\alpha Z - \beta$ P-a.s with some $\alpha, \beta \in \mathbb{R}_+$, then $\mathsf{E_Q} X^- < \infty$. By the definition of $\mathcal{R}$, $\mathsf{E_Q} X^+ \leq \mathsf{E_Q} X^-$. As a result, $\mathsf{E_Q}|X| < \infty$ and $\mathsf{E_Q} X \leq 0$. □

The following basic assumption is satisfied in all the particular models considered below.

Assumption 2.11. There exists $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$ (in particular, both sets might be empty).

Theorem 2.12 (FTAP). *Suppose that Assumption 2.11 is satisfied. Then the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition if and only if there exists an equivalent risk-neutral measure.*

The proof is based on a well known result of Kreps [K81] and Yan [Y80] (its proof can also be found in [S92], [S90], and other papers):

Lemma 2.13 (Kreps, Yan). *Let C be a $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone in L^∞ such that $C \supseteq L^\infty_-$ (L^∞_- is the set of negative elements of L^∞) and $C \cap L^0_+ = \{0\}$. Then there exists a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E_Q} X \leq 0$ for any $X \in C$.*

`Proof of Theorem 2.12.` *Step 1.* Let us prove the "only if" implication. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Lemma 2.13 applied to the $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone $A_4(Z_0)$ yields a probability measure $\mathsf{Q}_0 \sim \mathsf{P}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \leq 0$ for any $X \in A_4(Z_0)$. By the Fatou lemma, for any $X \in A$ such that $\frac{X}{Z_0 + \gamma(Z_0)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_0 + \gamma(Z_0)} \leq 0$ (note that $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_0 + \gamma(Z_0)} \wedge c \leq 0$ for any $c > 0$). Consider the probability measure $\mathsf{Q} = \frac{c}{Z_0 + \gamma(Z_0)} \mathsf{Q}_0$, where c is the normalizing constant (it exists since $Z_0 + \gamma(Z_0) \geq 1$). Then $\mathsf{Q} \in \mathcal{R}(Z_0) = \mathcal{R}$.

Step 2. Let us prove the "if" implication. Take $\mathsf{Q} \in \mathcal{R}$ and $Z \in B$. It follows from the Fatou lemma that Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where c is the normalizing constant. For any $X \in A$ such that $\frac{X}{Z + \gamma(Z)}$ is bounded below by a constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E_Q} X^- \leq \mathsf{E_Q}(\alpha Z + \alpha\gamma(Z)) < \infty,$$

and consequently,

$$\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{X}{Z + \gamma(Z)} = c\mathsf{E_Q} X \leq 0.$$

Hence, $\mathsf{E}_{\widetilde{\mathsf{Q}}} X \leq 0$ for any $X \in A_4(Z)$. As a result, $A_4(Z) \cap L^0_+ = \{0\}$. □

It is seen from the above proof that the implication $\mathcal{R} \neq \emptyset \Rightarrow$ NGA is true without Assumption 2.11. The following example shows that this assumption is essential for the reverse implication.

Example 2.14. Let $(X_t)_{t \in [0,1]}$ be a collection of independent Gaussian random variables with mean 1 and variance 1 defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $\mathcal{F} = \sigma(X_t; t \in [0,1])$ and

$$A = \left\{ \sum_{n=1}^{N} h_n X_{t_n} : N \in \mathbb{N},\ t_n \in [0,1],\ h_n \in \mathbb{R} \right\}.$$

Clearly, the only element of A that is bounded below is 0. This implies that the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition.

Suppose now that there exists an equivalent risk-neutral measure Q. Set $\rho = \frac{d\mathsf{Q}}{d\mathsf{P}}$. Note that $\mathcal{F} = \cup_C \sigma(X_t; t \in C)$, where the union is taken over all the countable sets $C \subset [0,1]$. Hence, there exists a countable set $C_0 \subset [0,1]$ such that ρ is $\sigma(X_t; t \in C_0)$-measurable. For any $t \notin C_0$, we have

$$\mathsf{E}_\mathsf{Q} X_t = \mathsf{E}_\mathsf{P} \rho X_t = \mathsf{E}_\mathsf{P} \rho \cdot \mathsf{E}_\mathsf{P} X_t = \mathsf{E}_\mathsf{P} X_t = 1.$$

As a result, there exists no equivalent risk-neutral measure. □

We conclude this section with an important notion, which allows one to build complicated models using simple ones.

Definition 2.15. A *combination* of arbitrage pricing models $(\Omega, \mathcal{F}, \mathsf{P}, A_\gamma)$, $\gamma \in \Gamma$ is the model $\left(\Omega, \mathcal{F}, \mathsf{P}, \sum_{\gamma \in \Gamma} A_\gamma\right)$, where

$$\sum_{\gamma \in \Gamma} A_\gamma := \left\{ \sum_{n=1}^{N} X_n : N \in \mathbb{N},\ X_n \in A_{\gamma_n},\ \gamma_n \in \Gamma \right\}.$$

The financial meaning of this definition is as follows. If A_γ is interpreted as the set of discounted incomes that can be obtained by trading assets from some collection $\mathcal{A}_\gamma$, then $\sum_{\gamma \in \Gamma} A_\gamma$ is the set of discounted incomes one can obtain by trading assets from $\cup_{\gamma \in \Gamma} \mathcal{A}_\gamma$.

2.3 Pricing of Contingent Claims

Let $(\Omega, \mathcal{F}, \mathsf{P}, A)$ be an arbitrage pricing model.

Definition 2.16. A *contingent claim* is a random variable F on $(\Omega, \mathcal{F}, \mathsf{P})$.

From the financial point of view, F is the payoff of the contingent claim discounted to the initial time.

Definition 2.17. (i) A real number x is a *fair price* of F if the combination $(\Omega, \mathcal{F}, \mathsf{P}, A + A(x))$, where

$$A(x) = \{h(F - x) : h \in \mathbb{R}\},$$

satisfies the NGA condition. (From the financial point of view, $A + A(x)$ is the set of discounted incomes that can be obtained by trading the "original" assets as well as buying or selling the contract F at the price x.) The set of fair prices of F will be denoted by $I(F)$.

(ii) A pair of real numbers (x, y) is a *fair bid-ask price* of F if the combination $(\Omega, \mathcal{F}, \mathsf{P}, A + A(x, y))$, where

$$A(x, y) = \{g(F - y) + h(x - F) : g, h \in \mathbb{R}_+\},$$

satisfies the NGA condition. (From the financial point of view, $A + A(x, y)$ is the set of discounted incomes that can be obtained by trading the "original" assets as well as selling the contract F at the price x or buying it at the price y.) The set of fair bid-ask prices of F will be denoted by $J(F)$.

(iii) The *lower* and *upper* prices of F are defined by

$$V_*(F) = \inf\{x : x \in I(F)\},$$
$$V^*(F) = \sup\{x : x \in I(F)\}.$$

Definition 2.18. Let C and D be two sets in $\mathbb{R}^d$. We will say that C and D are *approximately equal* if their interiors coincide and their closures coincide. This will be denoted as $C \approx D$.

For example, if C is a one-point set, then the approximate equality $D \approx C$ means that $D = C$; if C is an interval on the real line, then the approximate equality $D \approx C$ means that D is an interval with the same endpoints as C (so that D and C coincide up to the endpoints).

Theorem 2.19 (Main theorem for pricing contingent claims). *Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies Assumption 2.11 and the NGA condition, while F is bounded below. Then*

$$I(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{R}\} \approx \Big[\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F,\ \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F\Big], \tag{2.9}$$

$$\begin{aligned} J(F) &= \{(x,y) : x \le \mathsf{E}_\mathsf{Q} F \le y \text{ for some } \mathsf{Q} \in \mathcal{R}\} \\ &\approx \Big\{(x,y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F\Big\}, \end{aligned} \tag{2.10}$$

$$V_*(F) = \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F, \tag{2.11}$$

$$V^*(F) = \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F. \tag{2.12}$$

The expectation $\mathsf{E}_\mathsf{Q} F$ here is taken in the sense of finite expectations, i.e. we consider only those Q, for which $\mathsf{E}_\mathsf{Q} F < \infty$ (in particular, if $\mathsf{E}_\mathsf{Q} F = \infty$ for any $\mathsf{Q} \in \mathcal{R}$, then $I(F) = J(F) = \emptyset$).

`Proof`. Equalities (2.9), (2.11), and (2.12) follow from (2.10). Furthermore, it is sufficient to prove only the first equality in (2.10).

Step 1. Let $(x, y) \in J(F)$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Set $Z_1 = Z_0 + (F - y)$. Then $Z_1 \in B'$, where B' is defined by (2.6) with A replaced by

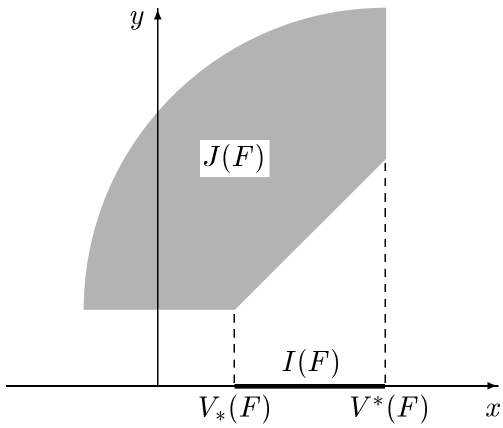
$$A' = \{X + g(F - y) + h(x - F) : X \in A,\ g, h \in \mathbb{R}_+\}.$$

Lemma 2.13 applied to the $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone $A'_4(Z_1)$ ($A'_4(Z_1)$ is defined by (2.7)) yields a probability measure $\mathsf{Q}_0 \sim \mathsf{P}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \le 0$ for any $X \in A'_4(Z_1)$. By the Fatou lemma, for any $X \in A'$ such that $\frac{X}{Z_1 + \gamma(Z_1)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_1+\gamma(Z_1)} \le 0$. Consider the probability measure $\mathsf{Q} = \frac{c}{Z_1+\gamma(Z_1)} \mathsf{Q}_0$, where c is the normalizing constant (it exists since $Z_1 + \gamma(Z_1) \ge 1$). Then $\mathsf{Q} \in \mathcal{R}(Z_1) \subseteq \mathcal{R}(Z_0) = \mathcal{R}$. Moreover, $\mathsf{E}_\mathsf{Q}(x - F) \le 0$ and $\mathsf{E}_\mathsf{Q}(F - y) \le 0$ since the random variables $\frac{x-F}{Z_1+\gamma(Z_1)}$ and $\frac{F-y}{Z_1+\gamma(Z_1)}$ are bounded below. Thus, $x \le \mathsf{E}_\mathsf{Q} F \le y$.

Step 2. Now, let (x, y) be a pair such that $x \le \mathsf{E}_\mathsf{Q} F \le y$ for some $\mathsf{Q} \in \mathcal{R}$. Take $Z \in B'$. Choose an arbitrary $Y = X + g(F - y) + h(x - F) \in A'$ (here $X \in A$) such that Y is bounded below. It follows from the condition $\mathsf{E_Q}F \le y$ that $\mathsf{E_Q}X^- < \infty$. As $\mathsf{Q} \in \mathcal{R}$, we have $\mathsf{E_Q}X \le 0$. This, combined with the condition $x \le \mathsf{E_Q}F \le y$, implies that $\mathsf{E_Q}Y \le 0$. By the Fatou lemma, Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where c is the normalizing constant. For any $Y = X + g(F - y) + h(x - F) \in A'$ (here $X \in A$) such that $\frac{Y}{Z+\gamma(Z)}$ is bounded below by some constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E_Q}Y^- \le \mathsf{E_Q}(\alpha Z + \alpha\gamma(Z)) < \infty.$$

Consequently, $\mathsf{E_Q}X^- < \infty$, $\mathsf{E_Q}X \le 0$, and $\mathsf{E_Q}Y \le 0$. This means that $\mathsf{E}_{\widetilde{\mathsf{Q}}}\frac{Y}{Z+\gamma(Z)} \le 0$. Hence, for any $Y \in A'_4(Z)$, we have $\mathsf{E}_{\widetilde{\mathsf{Q}}}Y \le 0$. This implies that $A'_4(Z) \cap L^0_+ = \{0\}$. As a result, $(x, y) \in J(F)$. □



`Figure 5.` The joint arrangement of $I(F)$, $J(F)$, $V_*(F)$, and $V^*(F)$

`Remarks`. `(i)` Theorem 2.19 remains valid if the condition "F is bounded below" is replaced by the condition "F is bounded above" (the proof remains the same).

`(ii)` Another way to define the lower and upper prices is through the sub- and superreplication, i.e.

$$C_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \le F \ \mathsf{P}\text{-a.s.}\}, \tag{2.13}$$

$$C^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \ge F \ \mathsf{P}\text{-a.s.}\}. \tag{2.14}$$

Obviously, under the assumptions of Theorem 2.19, we have

$$C_*(F) \le V_*(F) \le V^*(F) \le C^*(F).$$

In some models (for example, the models of Sections 3.1, 3.2), we have $C_*(F) = V_*(F)$, $C^*(F) = V^*(F)$. However, in the general case these equalities might be violated (see Example 3.27).

2.4 Pricing of Controlled Contingent Claims

Let $(\Omega, \mathcal{F}, \mathsf{P}, A)$ be an arbitrage pricing model.

Definition 2.20. A *controlled contingent claim* is a collection $(F_\lambda)_{\lambda\in\Lambda}$ of random variables on $(\Omega, \mathcal{F}, \mathsf{P})$.

From the financial point of view, Λ is the set of controls that are available to one party and F_λ is the payoff discounted to the initial time that this party obtains if he or she chooses the control λ.

The examples below show that various financial contracts can be represented as controlled contingent claims.

Example 2.21 (American and Bermudian options). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathsf{P})$ be a filtered probability space. Let $(C_t)_{t \in \mathbb{R}_+}$ be an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted process. From the financial point of view, C_t is the amount received by the holder of an (American or Bermudian) option if the option is exercised at time t. Let $r \in \mathbb{R}_+$ be the risk-free interest rate (we assume that it is constant).

An *American option* with maturity T (for the financial description, see [H97; Sect. 1.3]) can be represented as

$$\begin{aligned} \Lambda &= \{\lambda : \lambda \text{ is a } [0, T]\text{-valued } (\mathcal{F}_t)\text{-stopping time}\}, \\ F_\lambda &= e^{-r\lambda} C_\lambda. \end{aligned}$$

From the financial point of view, λ is the time, at which the option is exercised.

A *Bermudian option* with possible exercise times $T_1, \dots, T_N$ (for the financial description, see [H97; Sect. 6.3]) can be represented as

$$\begin{aligned} \Lambda &= \{\lambda : \lambda \text{ is a } \{T_1, \dots, T_n\}\text{-valued } (\mathcal{F}_t)\text{-stopping time}\}, \\ F_\lambda &= e^{-r\lambda} C_\lambda. \end{aligned}$$

From the financial point of view, λ is the time, at which the option is exercised. □

Example 2.22 (Convertible, puttable, and callable bonds). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathsf{P})$ be a filtered probability space. Let B be an $\mathbb{R}_+$-valued random variable on $(\Omega, \mathcal{F}, \mathsf{P})$. Let $(S_t)_{t \in [0,T]}$ and $(B_t)_{t \in [0,T]}$ be $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted processes. From the financial point of view, T is the maturity of a zero-coupon (convertible, puttable, or callable) bond; B is the amount received by a bondholder at time T if the bond is never converted (this concerns convertible bonds); S_t is the price at time t of a stock, into which the bond can be converted (this concerns convertible bonds); B_t is the amount received by a bondholder if the bond is redeemed at time t (this concerns puttable and callable bonds). Let $r \in \mathbb{R}_+$ be the risk-free interest rate.

A *convertible bond* (for the financial description, see [H97; Sect. 6.11]) can be represented as

$$\begin{aligned} \Lambda &= \{\lambda : \lambda \text{ is a } [0, T] \cup \{\infty\}\text{-valued } (\mathcal{F}_t)\text{-stopping time}\}, \\ F_\lambda &= e^{-r\lambda} S_\lambda I(\lambda \le T) + e^{-rT} B I(\lambda = \infty). \end{aligned}$$

From the financial point of view, λ is the time, at which the bond is converted into the stock (the event $\lambda = \infty$ means that the bond is never converted).

A *puttable bond* (for the financial description, see [H97; Sect. 16.2]) can be represented as

$$\begin{aligned} \Lambda &= \{\lambda : \lambda \text{ is a } [0, T]\text{-valued } (\mathcal{F}_t)\text{-stopping time}\}, \\ F_\lambda &= e^{-r\lambda} B_\lambda. \end{aligned}$$

From the financial point of view, λ is the time, at which the bond is presented for the redemption by the holder.

A *callable bond* (for the financial description, see [H97; Sect. 16.2]) can be represented as

$$\Lambda = \{\lambda : \lambda \text{ is a } [0,T]\text{-valued } (\mathcal{F}_t)\text{-stopping time}\},$$
$$F_\lambda = -e^{-r\lambda} B_\lambda.$$

From the financial point of view, λ is the time, at which the bond is redeemed by the issuer. The sign "$-$" reflects the fact that in this case the issuer (not the holder) of the bond chooses the control, so that F_λ is the amount "received" by the issuer if he or she chooses the control λ. A fair price of the callable bond would be minus the fair price of $(F_\lambda)_{\lambda\in\Lambda}$. □

Example 2.23 (Extendable and puttable interest-rate swaps). Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{Z}_+}, \mathsf{P})$ be a filtered probability space. Let $(\rho_n)_{n\in\mathbb{Z}_+}$ be an $\mathbb{R}_+$-valued $(\mathcal{F}_n)$-adapted random sequence. From the financial point of view, ρ_n is the instantaneous risk-free interest rate at time n. Let $r(n)$, $n \in \mathbb{Z}_+$ be the risk-free interest rate for the period $[0,n]$, i.e. $(1+r(n))^{-n}$ is the price at time 0 of a risk-free zero-coupon bond with maturity n and face value 1.

An *extendable interest-rate swap* with lifetime N and fixed rate ρ (for the financial description, see [H97; Sect. 5.6]) can be represented as

$$\Lambda = \{\lambda : \lambda \text{ is an } \{N, N+1, \dots\}\text{-valued } (\mathcal{F}_n)\text{-stopping time}\},$$
$$F_\lambda = \pm \sum_{n=1}^{\lambda} (1+r(n))^{-n}(\rho_n - \rho).$$

The choice of the sign here depends on whether one or another party has the right to extend the swap (compare with the previous example). From the financial point of view, λ is the time, up to which the swap is extended.

A *puttable interest-rate swap* (for the financial description, see [H97; Sect. 5.6]) can be represented as

$$\Lambda = \{\lambda : \lambda \text{ is a } \{0, \dots, N\}\text{-valued } (\mathcal{F}_n)\text{-stopping time}\},$$
$$F_\lambda = \pm \sum_{n=1}^{\lambda} (1+r(n))^{-n}(\rho_n - \rho).$$

From the financial point of view, λ is the time, at which the swap is terminated. □

Definition 2.24. (i) A real number x is a *fair price* of $(F_\lambda)_{\lambda\in\Lambda}$ if there exists $\lambda_0 \in \Lambda$ such that the combination $(\Omega, \mathcal{F}, \mathsf{P}, A + A(x, \lambda_0))$, where

$$A(x, \lambda_0) = \left\{ \sum_{n=1}^{N} h_n(F_{\lambda_n} - x) + h_0(x - F_{\lambda_0}) : N \in \mathbb{N},\ \lambda_n \in \Lambda,\ h_n \in \mathbb{R}_+,\ h_0 \in \mathbb{R}_+ \right\},$$

satisfies the NGA condition. (From the financial point of view, $A + A(x, \lambda_0)$ is the set of discounted incomes that can be obtained by trading the "original" assets, buying the contract $(F_\lambda)_{\lambda\in\Lambda}$ at the price x and using various controls as well as selling the contract $(F_\lambda)_{\lambda\in\Lambda}$ at the price x given that the buyer will choose the control λ_0.) The set of fair prices of $(F_\lambda)_{\lambda\in\Lambda}$ will be denoted by $I(F_\lambda; \lambda \in \Lambda)$.

(ii) A pair of real numbers (x,y) is a *fair bid-ask price* of $(F_\lambda;\lambda\in\Lambda)$ if there exists $\lambda_0\in\Lambda$ such that the combination $(\Omega,\mathcal{F},\mathsf{P},A+A(x,y,\lambda_0))$, where

$$A(x,y,\lambda_0)=\left\{\sum_{n=1}^N h_n(F_{\lambda_n}-y)+h_0(x-F_{\lambda_0}):N\in\mathbb{N},\ \lambda_n\in\Lambda,\ h_n\in\mathbb{R}_+,\ h_0\in\mathbb{R}_+\right\},$$

satisfies the NGA condition. The set of fair bid-ask prices of $(F_\lambda)_{\lambda\in\Lambda}$ will be denoted by $J(F_\lambda;\lambda\in\Lambda)$.

(iii) The *lower* and *upper* prices of $(F_\lambda)_{\lambda\in\Lambda}$ are defined by

$$V_*(F_\lambda;\lambda\in\Lambda)=\inf\{y:(x,y)\in J(F_\lambda;\lambda\in\Lambda)\},$$
$$V^*(F_\lambda;\lambda\in\Lambda)=\sup\{x:(x,y)\in J(F_\lambda;\lambda\in\Lambda)\}.$$

Remark. Let us discuss the financial meaning of the definition given above. If $x\in I(F_\lambda;\lambda\in\Lambda)$, then there exists $\lambda_0\in\Lambda$ with the property: if the buyer of $(F_\lambda)_{\lambda\in\Lambda}$ chooses this control, then there is no generalized arbitrage opportunity. So, in that case x is indeed a "fair" price. Suppose now that $x\notin I(F_\lambda;\lambda\in\Lambda)$. If we assume that the seller of $(F_\lambda)_{\lambda\in\Lambda}$ knows from the outset the control that the buyer will choose (for example, if $(F_\lambda)_{\lambda\in\Lambda}$ is an American option, then this assumption means that the seller knows the stopping time λ, at which the buyer will exercise the option, and not the value $\lambda(\omega)$ of this stopping time), then the seller has a generalized arbitrage opportunity. So, in that case x is not a "fair" price.

Important remark. Note that under the conditions of Theorem 2.19, we have

$$\inf\{x:x\in I(F)\}=\inf\{y:(x,y)\in J(F)\},$$
$$\sup\{x:x\in I(F)\}=\sup\{x:(x,y)\in J(F)\}$$

(they follow from (2.10)). Therefore, we may equivalently define the lower and upper prices of a contingent claim through the right-hand sides of these equalities. On the other hand, such equalities do not hold for a controlled contingent claim (see Example 2.26), and the reasonable definition of the lower and upper prices is the one given above.

Theorem 2.25 (Main theorem for pricing controlled contingent claims). *Suppose that the model $(\Omega,\mathcal{F},\mathsf{P},A)$ satisfies Assumption 2.11 and the NGA condition, while F_λ is bounded below for any $\lambda\in\Lambda$. Then*

$$I(F_\lambda;\lambda\in\Lambda)\subseteq\left[\inf_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E_Q}F_\lambda,\ \sup_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E_Q}F_\lambda\right],\tag{2.15}$$

$$J(F_\lambda;\lambda\in\Lambda)\approx\left\{(x,y):x\le y,\ x\le\sup_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E_Q}F_\lambda,\ y\ge\inf_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E_Q}F_\lambda\right\},\tag{2.16}$$

$$V_*(F_\lambda;\lambda\in\Lambda)=\inf_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E_Q}F_\lambda,\tag{2.17}$$

$$V^*(F_\lambda;\lambda\in\Lambda)=\sup_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E_Q}F_\lambda.\tag{2.18}$$

The supremum $\sup_{\lambda\in\Lambda}\mathsf{E_Q}F_\lambda$ is taken here as an $\mathbb{R}$-valued supremum, i.e. we consider only those Q, for which $\sup_{\lambda\in\Lambda}\mathsf{E_Q}F_\lambda<\infty$ (in particular, if $\sup_{\lambda\in\Lambda}\mathsf{E_Q}F_\lambda=\infty$ for any $\mathsf{Q}\in\mathcal{R}$, then $I(F_\lambda;\lambda\in\Lambda)=J(F_\lambda;\lambda\in\Lambda)=\emptyset$).

Proof. We will check only (2.16). Inclusion (2.15) is verified similarly, while equalities (2.17) and (2.18) follow from (2.16).

Step 1. Let us prove the inclusion

$$J(F_\lambda;\lambda\in\Lambda)\subseteq\Big\{(x,y): x\le y,\ x\le\sup_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E}_\mathsf{Q}F_\lambda,\ y\ge\inf_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E}_\mathsf{Q}F_\lambda\Big\}. \tag{2.19}$$

Let $(x,y)\in J(F_\lambda;\lambda\in\Lambda)$. Let λ_0 be an element of Λ such that the model $(\Omega,\mathcal{F},\mathsf{P},A+A(x,y,\lambda_0))$ satisfies the NGA condition. Take $Z_0\in B$ such that $\mathcal{R}=\mathcal{R}(Z_0)$. Set $Z_1=Z_0+(F_{\lambda_0}-y)$. Applying the same reasoning as in the proof of Theorem 2.19 (Step 1), we find a measure $\mathsf{Q}\in\mathcal{R}$ such that $\mathsf{E}_\mathsf{Q}(x-F_{\lambda_0})\le 0$ and $\mathsf{E}_\mathsf{Q}(F_\lambda-y)\le 0$ for any $\lambda\in\Lambda$. Then

$$x\le\mathsf{E}_\mathsf{Q}F_{\lambda_0}\le\sup_{\lambda\in\Lambda}\mathsf{E}_\mathsf{Q}F_\lambda\le y,$$

which means that (x,y) belongs to the right-hand side of (2.19).

Step 2. Let us prove the inclusion

$$\Big\{(x,y): x\le y,\ x\le\sup_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E}_\mathsf{Q}F_\lambda,\ y\ge\inf_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E}_\mathsf{Q}F_\lambda\Big\}^\circ\subseteq J(F_\lambda;\lambda\in\Lambda), \tag{2.20}$$

where "$\circ$" denotes the interior. Let (x,y) belong to the left-hand side of (2.20), i.e.

$$x<y,\qquad x<\sup_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E}_\mathsf{Q}F_\lambda,\qquad y>\inf_{\mathsf{Q}\in\mathcal{R}}\sup_{\lambda\in\Lambda}\mathsf{E}_\mathsf{Q}F_\lambda.$$

We can find measures $\mathsf{Q}_1,\mathsf{Q}_2\in\mathcal{R}$ such that

$$\sup_{\lambda\in\Lambda}\mathsf{E}_{\mathsf{Q}_1}F_\lambda>x,\qquad\sup_{\lambda\in\Lambda}\mathsf{E}_{\mathsf{Q}_2}F_\lambda<y.$$

Set $\mathsf{Q}(\alpha)=(1-\alpha)\mathsf{Q}_1+\alpha\mathsf{Q}_2$, $\alpha\in[0,1]$. Since the map $\alpha\mapsto\sup_{\lambda\in\Lambda}\mathsf{E}_{\mathsf{Q}(\alpha)}F_\lambda$ is continuous in α, there exists $\alpha_0\in(0,1)$ such that

$$x<\sup_{\lambda\in\Lambda}\mathsf{E}_{\mathsf{Q}(\alpha_0)}F_\lambda<y.$$

Note that $\mathsf{Q}(\alpha_0)\in\mathcal{R}$ due to the convexity of $\mathcal{R}$. Find $\lambda_0\in\Lambda$ such that $\mathsf{E}_{\mathsf{Q}(\alpha_0)}F_{\lambda_0}>x$. Applying the same reasoning as in the proof of Theorem 2.19 (Step 2), we verify that the model $(\Omega,\mathcal{F},\mathsf{P},A+A(x,y,\lambda_0))$ satisfies the NGA condition, which means that $(x,y)\in J(F_\lambda;\lambda\in\Lambda)$. □

The following example shows that for controlled contingent claims it could be more reasonable to define the set of fair prices of $(F_\lambda)_{\lambda\in\Lambda}$ not as $I(F_\lambda;\lambda\in\Lambda)$, but rather as the interval with the endpoints $\inf_{\mathsf{Q}\in\mathcal{R}}\mathsf{E}_\mathsf{Q}F_\lambda$ and $\sup_{\mathsf{Q}\in\mathcal{R}}\mathsf{E}_\mathsf{Q}F_\lambda$.

Example 2.26. Let $(\Omega,\mathcal{F},\mathsf{P},A)$ be an arbitrage pricing model that satisfies Assumption 2.11 and the NGA condition. Consider the controlled contingent claim defined by $\Lambda=(0,1)$, $F_\lambda=\lambda$. Then $I(F_\lambda;\lambda\in\Lambda)=\emptyset$. On the other hand, by Theorem 2.25,

$$\begin{aligned}J(F_\lambda;\lambda\in\Lambda)&\approx\{(x,y): x\le y,\ x\le 1,\ y\ge 1\},\\ V_*(F_\lambda;\lambda\in\Lambda)&=1,\\ V^*(F_\lambda;\lambda\in\Lambda)&=1.\end{aligned}$$

These values agree with the common sense as in this model it is reasonable to consider 1 as the "fair" price of $(F_\lambda)_{\lambda\in\Lambda}$. □

3 Particular Models with No Friction

In this chapter, the general approach introduced above is "projected" on various particular models with no friction. Each section is organized as follows. First, we describe a particular model and show how it can be embedded into the framework of the general arbitrage pricing model. In order to do this, we should only specify the set A of attainable incomes. Next, we provide a simple description of the set $\mathcal{R}$ of risk-neutral measures and prove that Assumption 2.11 is satisfied (we call the corresponding statement the Key Lemma of the section). Then

- Theorem 2.12 yields the necessary and sufficient conditions for the absence of the generalized arbitrage;
- Theorem 2.19 yields the form of fair prices of a contingent claim (that is bounded below);
- Theorem 2.25 yields the form of fair prices of a controlled contingent claim (that satisfies the conditions of this theorem).

3.1 One-Period Model

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Let $S_0 \in \mathbb{R}_+^d$ and S_1 be an $\mathbb{R}_+^d$-valued random vector on $(\Omega, \mathcal{F}, \mathsf{P})$ (note the difference as compared with Section 2.1, where we assumed that $S_0 \in \mathbb{R}^d$ and $S_1 : \Omega \to \mathbb{R}^d$). From the financial point of view, S_n^i is the price of the i-th asset at time n (assets $1, \dots, d$ are the same as in Section 2.1). Define $\overline{S}$ by (2.1). Define the set of attainable incomes by

$$A = \left\{ \sum_{i=1}^{d} h^i \big(\overline{S}_1^i - \overline{S}_0^i\big) : h^i \in \mathbb{R} \right\}.$$

`Notation`. Set $\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \mathsf{E}_\mathsf{Q} \overline{S}_1 = \overline{S}_0\}$.

Key Lemma 3.1. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}\bigg(\sum_{i=1}^{d} \big(\overline{S}_1^i - \overline{S}_0^i\big) \bigg) = \mathcal{M}.$$

This statement is clear (take Lemma 2.10 into account).

`Remarks.` `(i)` It follows from Theorem 2.4, Theorem 2.12, and Key Lemma 3.1 that in this model the NGA condition is equivalent to the NA condition.

(ii) Let $F \in L^0$ be bounded below. It follows from the previous remark that the objects $I(F)$, $V_*(F)$, and $V^*(F)$ introduced in Definition 2.17 coincide in this model with the objects $I(F)$, $V_*(F)$, and $V^*(F)$ introduced in Definition 2.5.

3.2 Multiperiod Model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_n)_{n=0,\dots,N}$ be an $\mathbb{R}_+^d$-valued $(\mathcal{F}_n)$-adapted random sequence. From the financial point of view, S_n^i is the price of the i-th asset at time n (assets $1, \dots, d$ are the same as in Section 2.1). Define $\overline{S}$ by (2.1) (we assume that the risk-free interest rate r and the dividend rates q^i are constant). Define the set of attainable incomes by

$$A = \left\{ \sum_{n=1}^{N} \sum_{i=1}^{d} H_n^i \big(\overline{S}_n^i - \overline{S}_{n-1}^i\big) : H_n^i \text{ is } \mathcal{F}_{n-1}\text{-measurable} \right\}. \tag{3.1}$$

Important remark. Note that A can equivalently be defined as

$$A = \left\{ \sum_{m=1}^{M} \sum_{i=1}^{d} H_m^i \big(\overline{S}_{v_m}^i - \overline{S}_{u_m}^i\big) : M \in \mathbb{N},\ u_m \le v_m \text{ are} \right.$$
$$\left. (\mathcal{F}_n)\text{-stopping times, } H_m^i \text{ is } \mathcal{F}_{u_m}\text{-measurable} \right\}.$$

Notation. Set $\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \overline{S} \text{ is an } (\mathcal{F}_n, \mathsf{Q})\text{-martingale}\}$.

Key Lemma 3.2. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}\left(\sum_{n=1}^{N} \sum_{i=1}^{d} \big(\overline{S}_n^i - \overline{S}_0^i\big) \right) = \mathcal{M}.$$

The proof employs the following statement (see [JS98] or [S99; Ch. II, § 1c]):

Lemma 3.3. *Let* $(X_n)_{n=0,\dots,N}$ *be an* $(\mathcal{F}_n)$*-local martingale such that* $\mathsf{E}|X_0| < \infty$ *and* $\mathsf{E_P} X_N^- < \infty$. *Then* X *is an* $(\mathcal{F}_n)$*-martingale.*

Proof of Key Lemma 3.2. Denote $\sum_{n=1}^N \sum_{i=1}^d \big(\overline{S}_n^i - \overline{S}_0^i\big)$ by Z_0.

Step 1. The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 2.10.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(Z_0)$. Fix $i \in \{1, \dots, d\}$, $n \in \{1, \dots, N\}$. For any $D \in \mathcal{F}_{n-1}$, we have

$$I_D\big(\overline{S}_n^i - \overline{S}_{n-1}^i\big) \ge -\overline{S}_{n-1}^i \ge -Z_0 + \alpha,$$
$$I_D\big(-\overline{S}_n^i + \overline{S}_{n-1}^i\big) \ge -\overline{S}_n^i \ge -Z_0 + \alpha$$

with some $\alpha \in \mathbb{R}_+$. By the definition of $\mathcal{R}(Z_0)$, $\mathsf{E_Q} I_D\big|\overline{S}_n^i - \overline{S}_{n-1}^i\big| < \infty$ and $\mathsf{E_Q} I_D\big(\overline{S}_n^i - \overline{S}_{n-1}^i\big) = 0$. This means that $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Fix

$$X = \sum_{n=1}^{N} \sum_{i=1}^{d} H_n^i \big(\overline{S}_n^i - \overline{S}_{n-1}^i\big) \in A.$$

The sequence

$$X_n = \sum_{m=1}^{n}\sum_{i=1}^{d} H_m^i(\overline{S}_m^i - \overline{S}_{m-1}^i), \quad n = 0,\dots,N$$

is an $(\mathcal{F}_n, \mathsf{Q})$-local martingale. It follows from Lemma 3.3 that $\mathsf{E}_\mathsf{Q} X^- \ge \mathsf{E}_\mathsf{Q} X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

Set $C_n(\omega) = \overline{\text{conv}}\,\text{supp}\,\text{Law}_\mathsf{P}(\overline{S}_{n+1}\,|\,\mathcal{F}_n)(\omega)$, $n = 0,\dots,N-1$ and let $C_n^\circ(\omega)$ denote the relative interior of $C_n(\omega)$. Then the FTAP in the discrete-time case (see [JS98] or [S99; Ch. V, § 2e]), Theorem 2.12, and Key Lemma 3.2 yield

Corollary 3.4 (FTAP). *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *the following conditions are equivalent:*

(a) *NGA;*
(b) *NA (i.e.* $A \cap L_+^0 = \{0\}$*);*
(c) $\overline{S}_n(\omega) \in C_n^\circ(\omega)$ *for any* $n = 0,\dots,N-1$ *and* P*-a.e.* ω*;*
(d) $\mathcal{M} \neq \emptyset$.

Remark. Let $F \in L^0$ be bounded below. It follows from the well-known results of financial mathematics (see [S99; Ch. VI, § 1c]) and Theorem 2.19 that if the model under consideration satisfies conditions (a)–(d) of Corollary 3.4, then $V_*(F) = C_*(F)$ and $V^*(F) = C^*(F)$, where $C_*(F)$ and $C^*(F)$ are defined by (2.13) and (2.14). In other words, our lower and upper prices coincide in this model with the traditional lower and upper prices.

3.3 Continuous-Time Model with a Finite Time Horizon

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_t)_{t\in[0,T]}$ be an $\mathbb{R}_+^d$-valued $(\mathcal{F}_t)$-adapted càdlàg process. From the financial point of view, S_t^i is the price of the i-th asset at time t (assets $1,\dots,d$ are the same as in Section 2.1). Let $r \in \mathbb{R}_+$ be the continuously compounded risk-free interest rate and $q^i \in \mathbb{R}_+$ be the continuously compounded dividend rate on the i-th asset in the case, where this asset is a dividend-paying stock, stock index, or a foreign currency. Define the *discounted price* of the i-th asset by

$$\overline{S}_t^i = \begin{cases} e^{-rt} S_t^i & \text{if the } i\text{-th asset is a traded asset providing no dividends,} \\ e^{(q^i - r)t} S_t^i & \text{if the } i\text{-th asset is a dividend-paying stock, stock index, or a foreign currency,} \\ S_t^i & \text{if the } i\text{-th asset is a futures.} \end{cases} \tag{3.2}$$

Define the set of attainable incomes by

$$A = \Big\{ \sum_{n=1}^{N}\sum_{i=1}^{d} H_n^i(\overline{S}_{v_n}^i - \overline{S}_{u_n}^i) : N \in \mathbb{N},\ u_n \le v_n \text{ are } (\mathcal{F}_t)\text{-stopping times, } H_n^i \text{ is } \mathcal{F}_{u_n}\text{-measurable} \Big\}. \tag{3.3}$$

From the financial point of view, A is the set of incomes discounted to time 0 that can be obtained by trading assets $1,\dots,d$ on the interval $[0,T]$ (and using the bank account to borrow/lend money).

`Notation`. Set $\mathcal{M}=\{\mathsf{Q}\sim\mathsf{P}:\overline{S}$ is an $(\mathcal{F}_t,\mathsf{Q})$-martingale$\}$.

Key Lemma 3.5. *For the model* $(\Omega,\mathcal{F},\mathsf{P},A)$, *we have*

$$\mathcal{R}=\mathcal{R}\biggl(\sum_{i=1}^{d}(\overline{S}_T^i-\overline{S}_0^i)\biggr)=\mathcal{M}.$$

`Proof`. Denote $\sum_{i=1}^{d}\bigl(\overline{S}_T^i-\overline{S}_0^i\bigr)$ by Z_0.

Step 1. The inclusion $\mathcal{R}\subseteq\mathcal{R}(Z_0)$ follows from Lemma 2.10.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0)\subseteq\mathcal{M}$. Take $\mathsf{Q}\in\mathcal{R}(Z_0)$. Fix $i\in\{1,\dots,d\}$. For any $u\in[0,T]$, the random variable $\overline{S}_u^i-\overline{S}_0^i$ is bounded below, and therefore, $\mathsf{E}_\mathsf{Q}\bigl(\overline{S}_u^i-\overline{S}_0^i\bigr)\le 0$. In particular, $\overline{S}_u^i$ is Q-integrable. For any $u\le v\in[0,T]$ and any $D\in\mathcal{F}_u$ such that $\overline{S}_u^i$ is bounded on D, the random variable $I_D\bigl(\overline{S}_v^i-\overline{S}_u^i\bigr)$ is bounded below, and hence, $\mathsf{E}_\mathsf{Q}I_D\bigl(\overline{S}_v^i-\overline{S}_u^i\bigr)\le 0$. This proves that $\overline{S}^i$ is an $(\mathcal{F}_t,\mathsf{Q})$-supermartingale. It follows from the definition of $\mathcal{R}(Z_0)$ that $\mathsf{E}_\mathsf{Q}\bigl(\overline{S}_T^i-\overline{S}_0^i\bigr)=0$. This implies that $\mathsf{Q}\in\mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M}\subseteq\mathcal{R}$. Take $\mathsf{Q}\in\mathcal{M}$. Fix

$$X=\sum_{n=1}^{N}\sum_{i=1}^{d}H_n^i\bigl(\overline{S}_{v_n}^i-\overline{S}_{u_n}^i\bigr)\in A.$$

Set

$$\begin{aligned}
C_t&=\sum_{n=1}^{N}[I(u_n\le t)+I(v_n\le t)],\quad t\in[0,T],\\
\sigma_k&=\inf\{t:C_t\ge k\},\quad k=0,\dots,2N,\\
G_t&=\sum_{n=1}^{N}H_nI(u_n<t\le v_n),\quad t\in[0,T],\\
X_t&=\int_0^t G_u d\overline{S}_u=\sum_{n=1}^{N}\sum_{i=1}^{d}H_n^i\bigl(\overline{S}_{v_n\wedge t}^i-\overline{S}_{u_n\wedge t}^i\bigr),\quad t\in[0,T].
\end{aligned}$$

Note that

$$\begin{aligned}
G_{\sigma_k}=&\sum_{n=1}^{N}H_nI(\sigma_{k-1}<\sigma_k,\ u_n\le\sigma_{k-1},\ v_n>\sigma_{k-1})\\
&+\sum_{n=1}^{N}H_nI(\sigma_{k-1}=\sigma_k,\ u_n<\sigma_{k-1},\ v_n\ge\sigma_{k-1}),\quad k=1,\dots,2N.
\end{aligned}$$

It is seen from this representation that G_{σ_k} is $\mathcal{F}_{\sigma_{k-1}}$-measurable. Since $G_t=G_{\sigma_k}$ for $t\in(\sigma_{k-1},\sigma_k]$, we have

$$X_{\sigma_k}-X_{\sigma_{k-1}}=\sum_{i=1}^{d}G_{\sigma_k}^i\bigl(\overline{S}_{\sigma_k}^i-\overline{S}_{\sigma_{k-1}}^i\bigr),\quad k=1,\dots,2N.$$

By the optional stopping theorem (see [RY99; Ch. II, Th. 3.2]), the sequence $(\overline{S}_{\sigma_k})_{k=0,\dots,2N}$ is an $(\mathcal{F}_{\sigma_k}, \mathsf{Q})$-martingale. Hence, the sequence $(X_{\sigma_k})_{k=0,\dots,2N}$ is an $(\mathcal{F}_{\sigma_k}, \mathsf{Q})$-local martingale. Note that $X_{\sigma_0} = 0$ and $X_{\sigma_{2N}} = X$. It follows from Lemma 3.3 that $\mathsf{E}_{\mathsf{Q}} X^- \geq \mathsf{E}_{\mathsf{Q}} X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

Remark. If the NGA condition is satisfied, then $\overline{S}$ is an $(\mathcal{F}_t, \mathsf{P})$-semimartingale (and hence, S is an $(\mathcal{F}_t, \mathsf{P})$-semimartingale). This follows from the fact that the semimartingale property is preserved under an equivalent change of measure (see [JS03; Ch. III, Th. 3.13]).

The approach to arbitrage pricing in the continuous-time setting proposed here differs considerably from the traditional approach. Let us briefly describe the latter one.

In the traditional approach, the price process S is assumed to be an $\mathbb{R}^d$-valued $(\mathcal{F}_t, \mathsf{P})$-semimartingale. The process $\overline{S}$ is defined by (3.2). The "set of attainable incomes" (although this term is not used in the traditional approach) has the form

$$A = \Big\{ \int_0^T H_u d\overline{S}_u : H = (H_t^1, \dots, H_t^d)_{t \in [0,T]} \text{ is an } (\mathcal{F}_t)\text{-predictable } \overline{S}\text{-integrable process satisfying the } \textit{admissibility} \text{ condition, i.e. there exists } a \in \mathbb{R} \text{ such that } \int_0^t H_u d\overline{S}_u \geq a \text{ for any } t \in [0,T] \Big\}. \quad (3.4)$$

(Here $\int_0^t H_u d\overline{S}_u$ is the vector stochastic integral; its definition can be found in [JS03; Ch. III, § 6c] or [SC02]). Consider the sets

$$\begin{aligned} A_1 &= \{X - Y : X \in A,\ Y \in L_+^0\}, \\ A_2 &= A_1 \cap L^\infty, \\ A_3 &= \text{closure of } A_2 \text{ in the norm topology of } L^\infty. \end{aligned}$$

The *no free lunch with vanishing risk* (NFLVR) condition is defined as: $A_3 \cap L_+^0 = \{0\}$.

The traditional FTAP (see [DS98], [K97]) states that a model satisfies the NFLVR condition if and only if there exists an equivalent *sigma-martingale measure*, i.e. a measure $\mathsf{Q} \sim \mathsf{P}$ such that $\overline{S}$ is an $(\mathcal{F}_t, \mathsf{Q})$-sigma-martingale. Recall that a process $(X_t)_{t \in [0,T]}$ is called a *sigma-martingale* if there exists a sequence of predictable sets $(D_n)_{n \in \mathbb{N}}$ such that $D_n \subseteq D_{n+1}$, $\bigcup_n D_n = \Omega \times [0,T]$, and for any n, the stochastic integral $\int_0^\cdot I_{D_n}(s) dX_s$ is a uniformly integrable martingale (this definition was proposed by Goll and Kallsen [GK03]; it is equivalent to the original definition of Chou [C79] and Émery [E80]). The class of sigma-martingales contains the class of local martingales and is wider as shown by the Émery example (see [E80]). However, an $\mathbb{R}_+^d$-valued sigma-martingale is necessarily a local martingale as shown by Ansel and Stricker [AS94].

The set of fair prices of a contingent claim F is defined as the interval with the endpoints $C_*(F)$ and $C^*(F)$, where

$$\begin{aligned} C_*(F) &= \sup\{x : \text{there exists } X \in A \text{ such that } x - X \leq F \ \mathsf{P}\text{-a.s.}\}, \\ C^*(F) &= \inf\{x : \text{there exists } X \in A \text{ such that } x + X \geq F \ \mathsf{P}\text{-a.s.}\} \end{aligned}$$

(here A is given by (3.4)). It follows from [DS98], [FK98], and [FK97] that if the NFLVR condition is satisfied and F is bounded below, then

$$C^*(F) = \sup_{\mathsf{Q} \in \mathcal{M}_\sigma} \mathsf{E}_{\mathsf{Q}} F, \quad (3.5)$$

	Traditional approach	**Proposed approach**
The price process	$\mathbb{R}^d$-valued semimartingale	$\mathbb{R}^d_+$-valued adapted càdlàg process
Trading strategies	Predictable strategies satisfying the integrability and the admissibility conditions	Simple strategies with no integrability and no admissibility conditions imposed
The variant of the no-arbitrage condition	NFLVR	NGA
FTAP	NFLVR $\Longleftrightarrow$ existence of an equivalent sigma-martingale measure	NGA $\Longleftrightarrow$ existence of an equivalent martingale measure
Set of fair prices of a contingent claim	$[C_*(F), C^*(F)]$	$I(F)$

Table 3. The differences between the traditional approach to asset pricing in the continuous-time setting and the proposed approach

where

$$\mathcal{M}_\sigma = \{\mathsf{Q} \sim \mathsf{P} : \overline{S} \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-sigma-martingale}\}. \tag{3.6}$$

Remarks. (i) Note that the attainable incomes provided by (3.3) can be represented as the stochastic integrals of piecewise constant processes with respect to $\overline{S}$, namely

$$\sum_{n=1}^{N}\sum_{i=1}^{d} H_n^i(\overline{S}_{v_n}^i - \overline{S}_{u_n}^i) = \int_0^T G_u d\overline{S}_u,$$

where

$$G_t = \sum_{n=1}^{N} H_n I(u_n < t \le v_n), \quad t \in [0,T]. \tag{3.7}$$

(ii) In practice one can realize only simple strategies, i.e. the strategies of the form (3.7). Thus, the definition of the free lunch with vanishing risk employs two limit procedures: first, passing from simple strategies to general predictable strategies (it is meant that one can approximate random variables of the form (3.4) by using simple strategies); second, passing from A given by (3.4) to A_3 (it is meant that one can approximate elements of A_3 by elements of A_1). As for the generalized arbitrage, it employs only one limit procedure, namely, passing from A given by (3.3) to $A_5(Z)$ given by (2.8) (it is meant that one can approximate elements of $A_5(Z)$ by elements of A_1).

(iii) Any martingale is a sigma-martingale. Thus, if a model is arbitrage-free in the proposed approach (i.e. it satisfies the NGA condition defined through simple strategies), then it is arbitrage-free in the traditional approach (i.e. it satisfies the NFLVR condition defined through predictable admissible strategies). The reverse statement is not true (see Example 3.9).

Let us now give 4 examples and 2 remarks, which illustrate the problems that arise when one applies the traditional approach.

The first two examples and the remark following them show that the admissibility condition leads to an inadmissible restriction of the class of strategies (by a strategy we mean a process H that appears in (3.4)).

Example 3.6. Consider the Black–Scholes model, i.e. $\overline{S}_t = e^{\mu t+\sigma B_t}$, $t \in [0,T]$, where B is a Brownian motion. Let $\mathcal{F}_t = \mathcal{F}_t^{\overline{S}}$, $\mathcal{F} = \mathcal{F}_T$. Then the strategy $H_t = -1$, $t \in [0,T]$ is not admissible. In other words, the admissibility condition prohibits in this model the strategy that consists in the short selling of the asset at time 0 and buying it back at time T. □

Example 3.7. Consider the exponential Lévy model, i.e. $\overline{S}_t = e^{X_t}$, $t \in [0,T]$, where X is a Lévy process. Let $\mathcal{F}_t = \mathcal{F}_t^{\overline{S}}$, $\mathcal{F} = \mathcal{F}_T$. Suppose that the jumps of X are not bounded from above (the majority of the exponential Lévy models used in modern financial mathematics satisfy this condition). One can check that if H is an admissible strategy, then $H(\omega,t) \geq 0$ $\mathsf{P} \times \mu_L$-a.e, where μ_L is the Lebesgue measure on $[0,T]$. In other words, the admissibility condition prohibits in this model all the strategies employing short selling. Clearly, this is an unacceptable restriction: for example, when hedging a put option in practice, one employs strategies H with $H < 0$ identically (for more details, see [H97; Ch. 14]). □

Remark. Another drawback of the admissibility condition is as follows. Such a condition is not imposed in the discrete-time models, but it is imposed in the continuous-time models. This leads to an unpleasant unbalance. In particular, when one embeds a discrete-time model $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P}, (\overline{S}_n)_{n=0,\dots,N})$ into a continuous-time model (this is done in the canonical way; see [JS03; Ch. I, § 1f]), then the set of attainable incomes defined for this continuous-time model by (3.4) does not coincide with the set of attainable incomes defined for the original discrete-time model by (3.1).

The next example shows that in some models the traditional interval of fair prices is too wide.

Example 3.8. Let $\overline{S}_t = I(t<T) + \xi I(t=T)$, $t \in [0,T]$, where ξ is an $\mathbb{R}_+$-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with the property: for any $a \in \mathbb{R}_{++}$, $\mathsf{P}(\xi < a) > 0$ and $\mathsf{P}(\xi > a) > 0$. Let $\mathcal{F}_t = \mathcal{F}_t^{\overline{S}}$, $\mathcal{F} = \mathcal{F}_T$. Consider $F = \overline{S}_T$.

Let us find $C_*(F)$. Let H be a predictable admissible strategy and $x \in \mathbb{R}$ be such that

$$x - \int_0^T H_u d\overline{S}_u \leq F. \tag{3.8}$$

Note that

$$\int_0^T H_u d\overline{S}_u = H_T \Delta \overline{S}_T = H_T(\xi - 1).$$

Since H is $(\mathcal{F}_t)$-predictable and $\mathcal{F}_t = \{\emptyset, \Omega\}$ for $t < T$, H_T is a real number. The admissibility condition, together with the property $\mathsf{P}(\xi > a) > 0$ for any $a \in \mathbb{R}_{++}$, shows that $H_T \geq 0$. This, combined with (3.8) and with the property $\mathsf{P}(\xi < a) > 0$ for any $a \in \mathbb{R}_{++}$, yields $x \leq 0$. Consequently, $C_*(F) = 0$.

In a similar way one checks that $C^*(F) = 1$. Thus, the interval of fair prices provided by the traditional approach is $[0,1]$. On the other hand, the interval of fair prices provided by common sense consists only of point 1 since F can be replicated by buying the asset (whose discounted price is given by $(\overline{S}_t)_{t\in[0,T]}$) at time 0. □

`Remark.` In the model of the previous example, we have, due to the result of Ansel and Stricker [AS94],

$$\mathcal{M}_\sigma = \{\mathsf{Q} \sim \mathsf{P} : \overline{S} \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-local martingale}\}$$

($\mathcal{M}_\sigma$ is given by (3.6)). Furthermore, for any $(\mathcal{F}_t)$-stopping time τ, we have either $\tau = T$ P-a.s. or $\tau < T$ P-a.s. Consequently,

$$\mathcal{M}_\sigma = \{\mathsf{Q} \sim \mathsf{P} : \overline{S} \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\} = \{\mathsf{Q} \sim \mathsf{P} : \mathsf{E}_\mathsf{Q}\xi = 1\}.$$

Therefore, $\inf_{\mathsf{Q}\in\mathcal{M}_\sigma} \mathsf{E}_\mathsf{Q} F = 1$. This shows that the equality $C_*(F) = \inf_{\mathsf{Q}\in\mathcal{M}_\sigma} \mathsf{E}_\mathsf{Q} F$, which is dual to (3.5), is not true for F bounded below. (One way to overcome this problem was proposed in [DS98]. Namely, the authors of that paper altered the definition of $C_*(F)$ and $C^*(F)$ by introducing the so-called w-admissibility condition as a substitute of the admissibility condition. However, an unpleasant feature of this definition is that it depends on the choice of a so-called weight function.)

The fourth example is the most striking one. It shows that the use of the traditional approach may lead to mispricing of contingent claims.

Example 3.9. Let $\overline{S}_t = |B_t|^{-1}$, $t \in [0,T]$, where B is a 3-dimensional Brownian motion started at a point $B_0 \neq 0$ on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $\mathcal{F}_t = \mathcal{F}_t^{\overline{S}}$, $\mathcal{F} = \mathcal{F}_T$. Without loss of generality, $B_0^2 = B_0^3 = 0$. Note that

$$\mathsf{E}_\mathsf{P}\overline{S}_T = \mathsf{E}_\mathsf{P}((B_T^1)^2 + (B_T^2)^2 + (B_T^3)^2)^{-1/2} \le \mathsf{E}_\mathsf{P}((B_T^2)^2 + (B_T^3)^2)^{-1/2} = \frac{const}{\sqrt{T}}.$$

We take T large enough, so that $\mathsf{E}_\mathsf{P}\overline{S}_T < \overline{S}_0$ (actually, $\mathsf{E}_\mathsf{P}\overline{S}_T < \overline{S}_0$ for any $T > 0$). Consider $F = \overline{S}_T$.

Let us find $C^*(F)$. Applying Itô's formula and P. Lévy's characterization theorem (see [RY99; Ch. IV, Th. 3.6]), we conclude that

$$\overline{S}_t = \overline{S}_0 + \int_0^t \overline{S}_u^2 dW_u, \quad t \in [0,T], \tag{3.9}$$

where W is an $(\mathcal{F}_t, \mathsf{P})$-Brownian motion. Furthermore, Itô's theorem (see [Ø03; Th. 5.2.1]) guarantees that $(\overline{S}, W)$ is a strong solution of SDE (3.9), i.e. $\mathcal{F}_t^{\overline{S}} \subseteq \mathcal{F}_t^W$. It is clear from (3.9) that $\mathcal{F}_t^W \subseteq \mathcal{F}_t^{\overline{S}}$, and hence, $\mathcal{F}_t^W = \mathcal{F}_t^{\overline{S}} = \mathcal{F}_t$. Set $F_t = \mathsf{E}_\mathsf{P}(F \,|\, \mathcal{F}_t)$, $t \in [0,T]$. By the representation theorem for the Brownian motion (see [RY99; Ch. V, Th. 3.5]), there exists an $(\mathcal{F}_t)$-predictable W-integrable process K such that

$$F_t = \mathsf{E}_\mathsf{P} F + \int_0^t K_u dW_u, \quad t \in [0,T].$$

In view of (3.9),

$$F_t = \mathsf{E}_\mathsf{P} F + \int_0^t \frac{K_u}{\overline{S}_u^2}\, d\overline{S}_u = \mathsf{E}_\mathsf{P} F + \int_0^t H_u d\overline{S}_u, \quad t \in [0,T]. \tag{3.10}$$

Since $F_t \geq 0$, the strategy H is admissible. Consequently, $C^*(F) \leq \mathsf{E}_\mathsf{P} F$.

Similarly, by considering $F_t^n = \mathsf{E}_\mathsf{P}(FI(F \leq n) \,|\, \mathcal{F}_t)$, we prove that $C_*(F) \geq \mathsf{E}_\mathsf{P} F$. As a result, the fair price provided by the traditional approach is $\mathsf{E}_\mathsf{P} F = \mathsf{E}_\mathsf{P} \overline{S}_T$. On the other hand, the fair price provided by common sense is $\overline{S}_0$, which is not equal to $\mathsf{E}_\mathsf{P} \overline{S}_T$! □

The problems described above do not arise in the approach proposed here.

Indeed, no admissibility restriction is imposed in this approach, which solves the problems described in Examples 3.6, 3.7, and the remark following Example 3.7. In particular, when one embeds a discrete-time model $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P}, (\overline{S}_n)_{n=0,\dots,N})$ into a continuous-time model, then the set of attainable incomes defined for this continuous-time model by (3.3) coincides with the set of attainable incomes defined for the original discrete-time model by (3.1).

In Example 3.8, we have, due to Theorem 2.19,

$$I(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{M}\} = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \sim \mathsf{P},\ \mathsf{E}_\mathsf{Q} \xi = 1\} = \{1\},$$

which agrees with common sense.

By Theorem 2.19, we have $V_*(F) = \inf_{\mathsf{Q} \in \mathcal{M}} \mathsf{E}_\mathsf{Q} F$ for any F bounded below, which solves the problem mentioned in the remark following Example 3.8.

Finally, in Example 3.9, P is the only local martingale measure for $\overline{S}$. Indeed, if $\mathsf{Q} \sim \mathsf{P}$ is a local martingale measure for $\overline{S}$, then $\overline{S}$ satisfies SDE (3.9) with respect to Q. By Itô's theorem (see [Ø03; Th. 5.2.1]), there are strong existence and pathwise uniqueness for this SDE, and the Yamada–Watanabe theorem (see [RY99; Ch. IX, Th. 1.7]) implies the uniqueness in law. Hence, $\mathsf{Q} = \mathsf{P}$. Since P is not a martingale measure, there exists no equivalent martingale measure. This means that the model considered in Example 3.9 does not satisfy the NGA condition, and the paradox is solved.

`Remark`. An "arbitrage opportunity" in the model of Example 3.9 can be constructed as follows. Consider the strategy $G = H - 1$, where H is given by (3.10). Then

$$\int_0^T G_u d\overline{S}_u = \int_0^T H_u d\overline{S}_u - \overline{S}_T + \overline{S}_0 = -\mathsf{E}_\mathsf{P} \overline{S}_T + \overline{S}_0 > 0.$$

The strategy G is not admissible, so it does not yield a free lunch with vanishing risk opportunity. It does not yield a generalized arbitrage opportunity either, but it can be used to construct a generalized arbitrage opportunity as follows. There exist simple strategies (i.e. strategies of the form (3.7)) $(\widetilde{H}_n)_{n \in \mathbb{N}}$ such that

$$\sup_{t \in [0,T]} \left| \int_0^t \widetilde{H}_{nu} d\overline{S}_u - \int_0^t H_u d\overline{S}_u \right| \xrightarrow[n \to \infty]{\mathsf{P}} 0.$$

Set

$$\tau_n = \inf\left\{ t \in [0,T] : \int_0^t \widetilde{H}_{nu} d\overline{S}_u \leq -\mathsf{E}_\mathsf{P} \overline{S}_T - 1 \right\},$$
$$H_{nt} = \widetilde{H}_{nt} I(t \leq \tau_n), \quad t \in [0,T].$$

Since

$$\int_0^t H_u d\overline{S}_u \geq -\mathsf{E}_\mathsf{P} \overline{S}_T, \quad t \in [0,T],$$

we get

$$\int_0^T H_{nu} d\overline{S}_u \xrightarrow[n\to\infty]{\mathsf{P}} \int_0^T H_u d\overline{S}_u = \overline{S}_T - \mathsf{E}_{\mathsf{P}}\overline{S}_T.$$

Set $G_n = H_n - 1$. Then, for $X_n = \int_0^T G_{nu} d\overline{S}_u$, we have $X_n \in A$, where A is given by (3.3). Furthermore, $X_n \geq -\overline{S}_T + \overline{S}_0 - \mathsf{E}_{\mathsf{P}}\overline{S}_T - 1$ P-a.s. for any $n \in \mathbb{N}$ and

$$X_n \xrightarrow[n\to\infty]{\mathsf{P}} \overline{S}_0 - \mathsf{E}_{\mathsf{P}}\overline{S}_T > 0.$$

Note that $Z := \overline{S}_T - \overline{S}_0$ belongs to B, where B is given by (2.6). Take $Y_n \in L^0_+$, $n \in \mathbb{N}$ such that

$$\frac{X_n - Y_n}{Z + \gamma(Z)} = \frac{X_n}{Z + \gamma(Z)} \wedge (\overline{S}_0 - \mathsf{E}_{\mathsf{P}}\overline{S}_T),$$

and then

$$\frac{X_n - Y_n}{Z + \gamma(Z)} \xrightarrow[n\to\infty]{\sigma(L^\infty, L^1(\mathsf{P}))} \frac{\overline{S}_0 - \mathsf{E}_{\mathsf{P}}\overline{S}_T}{Z + \gamma(Z)}.$$

This yields a generalized arbitrage opportunity in the model of Example 3.9.

One of the problems associated with the model under consideration is related to the change of numéraire. It is as follows. Let $S = (S^0_t, \dots, S^d_t)_{t\in[0,T]}$ be the price process of $d+1$ assets. We assume that each its component is strictly positive. Let us choose the 0-th asset as a numéraire, i.e. we define the discounted price process as $\overline{S}^i = S^i/S^0$, $i = 0, \dots, d$ and define the set of attainable incomes by (3.2) or (3.4), depending on the choice of the approach. (In order to embed the model of the present section into this framework, one should take $S^0_t = e^{rt}$ and appropriately define S^i_t by multiplying the true price process of the i-th asset by a factor $e^{q^i t}$ if the i-th asset is a dividend-paying stock, stock index or a foreign currency or by a factor e^{rt} if the i-th asset is a futures.) Now let us choose another asset as a numéraire (for example, the 1-st asset), i.e we define the new discounted process $\widetilde{S}$ as $\widetilde{S}^i = S^i/S^1$, $i = 0, \dots, d$ and define the set of attainable incomes $\widetilde{A}$ through $\widetilde{S}$. The problem is whether the NFLVR or NGA property holds or does not hold for the models $(\Omega, \mathcal{F}, \mathsf{P}, A)$ and $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ simultaneously.

In the traditional approach, the answer is negative, as shown by the example below (it is borrowed from [DS95]). Let us mention in this connection the papers [DS95'] and [DS96] devoted to the study, under which additional assumptions the NFLVR property is preserved under the change of numéraire.

Example 3.10. Let $S^0 = 1$ and $S^1 = |B|^{-1}$, where B is a 3-dimensional Brownian motion started at a point $B_0 \neq 0$. Let $\mathcal{F}_t = \mathcal{F}^S_t$, $\mathcal{F} = \mathcal{F}_T$. The model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ (A is defined by (3.4)) satisfies the NFLVR condition since the process $\overline{S}$ is a local martingale with respect to the original probability measure (see representation (3.9)). On the other hand, $\widetilde{S}^0 = |B|$ (this is a 3-dimensional Bessel process) and $\widetilde{S}^1 = 1$. If Q is an equivalent sigma-martingale measure for $\widetilde{S}$, then, by the result of Ansel and Stricker [AS94], $\widetilde{S}^0$ is an $(\mathcal{F}_t, \mathsf{Q})$-local martingale. Using Itô's formula, one easily checks that the quadratic variation of $\widetilde{S}^0$ is given by $[\widetilde{S}^0]_t = t$. P. Lévy's characterization theorem (see [RY99; Ch. IV, Th. 3.8]) now implies that $\widetilde{S}^0$ is a Q-Brownian motion. But this contradicts the positivity of $\widetilde{S}^0$. Hence, the model $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ does not satisfy the NFLVR condition. □

In contrast, the change of numéraire preserves the NGA property, as shown by the statement below.

Theorem 3.11 (Change of numéraire). *Let A (resp., $\widetilde{A}$) be defined through $\overline{S}$ (resp., $\widetilde{S}$) by (3.3). Then the models $(\Omega, \mathcal{F}, \mathsf{P}, A)$ and $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ satisfy or do not satisfy the NGA condition simultaneously.*

The proof employs the following statement (see [JS03; Ch. III, Prop. 3.8]).

Lemma 3.12. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space and $\mathsf{Q} \ll \mathsf{P}$. Let $Z_t = \frac{d\mathsf{Q}|\mathcal{F}_t}{d\mathsf{P}|\mathcal{F}_t}$, $t \in [0,T]$ be the density process of Q with respect to P. Then a process M is an $(\mathcal{F}_t, \mathsf{Q})$-martingale if and only if MZ is an $(\mathcal{F}_t, \mathsf{P})$-martingale.*

`Proof of Theorem 3.11.` Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition. Then there exists a probability measure $\overline{\mathsf{Q}} \sim \mathsf{P}$ such that $\overline{S}$ is an $(\mathcal{F}_t, \overline{\mathsf{Q}})$-martingale. Let $\overline{Z}$ denote the density process of $\overline{\mathsf{Q}}$ with respect to P. Consider the process $\widetilde{Z} = c\overline{S}^1\overline{Z}$, where the constant c is chosen in such a way that $\widetilde{Z}_0 = 1$. As $\overline{S}^1$ is an $(\mathcal{F}_t, \overline{\mathsf{Q}})$-martingale, then, by Lemma 3.12, $\widetilde{Z}$ is an $(\mathcal{F}_t, \mathsf{P})$-martingale. Hence, $\widetilde{Z}$ is the density process of a probability measure $\widetilde{\mathsf{Q}} = \widetilde{Z}_T\mathsf{P}$ with respect to P (note that $\widetilde{\mathsf{Q}} \sim \mathsf{P}$ since $\overline{S}^1$ and $\overline{Z}$ are strictly positive). As $\overline{S}$ is an $(\mathcal{F}_t, \overline{\mathsf{Q}})$-martingale, then, by Lemma 3.12, the process $\widetilde{S}\widetilde{Z} = c\overline{S}\,\overline{Z}$ is an $(\mathcal{F}_t, \mathsf{P})$-martingale, which (again by Lemma 3.12) implies that $\widetilde{S}$ is an $(\mathcal{F}_t, \widetilde{\mathsf{Q}})$-martingale. Hence, the model $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ satisfies the NGA condition. □

To conclude the section, we show that the proposed approach to pricing in the continuous-time setting agrees with the Black–Scholes formula (see [BS73]) as well as with its extension to dividend-paying stocks provided by Merton (see [M73]) and the extension to futures prices provided by Black (see [B76]).

Example 3.13 (Black–Scholes model). Let the price of a traded asset providing no dividends have the form $S_t = S_0 e^{\mu t + \sigma B_t}$, $t \in [0,T]$, where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R} \setminus \{0\}$, and B is a Brownian motion. Let $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F} = \mathcal{F}_T$. The discounted price process $\overline{S}$ has the form $\overline{S}_t = e^{-rt}S_t$, where r is the risk-free interest rate. By Theorem 2.12, this model satisfies the NGA condition. Let $F \in L^0$ be bounded below and Q-integrable, where Q is the unique martingale measure for $\overline{S}$. According to Theorem 2.19,

$$I(F) = \{\mathsf{E}_\mathsf{Q} F\}, \qquad V_*(F) = \mathsf{E}_\mathsf{Q} F, \qquad V^*(F) = \mathsf{E}_\mathsf{Q} F. \tag{3.11}$$

In particular, for $F = e^{-rT}(S_T - K)^+$ (recall that F denotes the discounted payoff), we arrive at the *Black–Scholes formula*:

$$\begin{aligned} V_*(F) = V^*(F) &= \mathsf{E}_\mathsf{Q} e^{-rT}(e^{rT}\overline{S}_T - K)^+ \\ &= S_0\Phi\Big(\frac{\ln S_0/K + (r+\sigma^2/2)T}{\sigma\sqrt{T}}\Big) - e^{-rT}K\Phi\Big(\frac{\ln S_0/K + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\Big), \end{aligned}$$

where Φ is the distribution function of the standard normal distribution. □

Example 3.14 (Merton's model). The framework is the same as in Example 3.13, but S now means the price of a dividend-paying stock, stock index, or a foreign currency with a dividend rate q. The discounted price process $\overline{S}$ has the form $\overline{S}_t = e^{(q-r)t}S_t$. By Theorem 2.12, this model satisfies the NGA condition. Let

$F \in L^0$ be bounded below and Q-integrable, where Q is the unique martingale measure for $\overline{S}$. According to Theorem 2.19, equalities (3.11) are true. In particular, for $F = e^{-rT}(S_T - K)^+$, we arrive at *Merton's formula*:

$$\begin{aligned} V_*(F) &= V^*(F) = \mathsf{E}_{\mathsf{Q}} e^{-rT}(e^{(r-q)T}\overline{S}_T - K)^+ \\ &= e^{-qT} S_0 \Phi\biggl(\frac{\ln S_0/K + (r-q+\sigma^2/2)T}{\sigma\sqrt{T}}\biggr) - e^{-rT} K \Phi\biggl(\frac{\ln S_0/K + (r-q-\sigma^2/2)T}{\sigma\sqrt{T}}\biggr). \end{aligned}$$
□

Example 3.15 (Black's model). The framework is the same as in Example 3.13, but S now means a futures price. The discounted price process $\overline{S}$ has the form $\overline{S}_t = S_t$. By Theorem 2.12, this model satisfies the NGA condition. Let $F \in L^0$ be bounded below and Q-integrable, where Q is the unique martingale measure for $\overline{S}$. According to Theorem 2.19, equalities (3.11) are true. In particular, for $F = e^{-rT}(S_T - K)^+$, we arrive at *Black's formula*:

$$\begin{aligned} V_*(F) &= V^*(F) = \mathsf{E}_{\mathsf{Q}} e^{-rT}(\overline{S}_T - K)^+ \\ &= e^{-rT} S_0 \Phi\biggl(\frac{\ln S_0/K + \sigma^2 T/2}{\sigma\sqrt{T}}\biggr) - e^{-rT} K \Phi\biggl(\frac{\ln S_0/K - \sigma^2 T/2}{\sigma\sqrt{T}}\biggr). \end{aligned}$$
□

3.4 Continuous-Time Model with the Infinite Time Horizon

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_t)_{t\in\mathbb{R}_+}$ be an $\mathbb{R}_+^d$-valued $(\mathcal{F}_t)$-adapted càdlàg process. From the financial point of view, S_t^i is the price of the i-th asset at time t (assets $1, \dots, d$ are the same as in Section 2.1). Define $\overline{S}$ by (3.2). Define the set of attainable incomes by

$$\begin{aligned} A = \biggl\{ &\sum_{n=1}^{N} \sum_{i=1}^{d} H_n^i (\overline{S}_{v_n}^i - \overline{S}_{u_n}^i) : N \in \mathbb{N},\ u_n \le v_n \text{ are } (\mathcal{F}_t)\text{-stopping times} \\ &\text{such that } \{v_n = \infty\} \subseteq \{\exists \lim_{t\to\infty} \overline{S}_t\},\ H_n^i \text{ is } \mathcal{F}_{u_n}\text{-measurable}\biggr\}. \end{aligned} \tag{3.12}$$

`Notation`. Set $\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \overline{S} \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-uniformly integrable martingale}\}$.

Key Lemma 3.16. *Suppose that the limit $\overline{S}_\infty = \lim_{t\to\infty} \overline{S}_t$ exists P-a.s. Then, for the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}\biggl(\sum_{i=1}^{d} (\overline{S}_\infty^i - \overline{S}_0^i)\biggr) = \mathcal{M}.$$

`Proof`. Note that $(\overline{S}_t)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t, \mathsf{Q})$-uniformly integrable martingale if and only if $(\overline{S}_t)_{t\in[0,\infty]}$ is a $(\mathcal{G}_t, \mathsf{Q})$-martingale, where

$$\mathcal{G}_t = \begin{cases} \mathcal{F}_t & \text{if } t \in \mathbb{R}_+, \\ \mathcal{F} & \text{if } t = \infty \end{cases}$$

(this statement follows from [RY99; Ch. II, Th. 3.1]). The desired statement can now be proved in the same way as Key Lemma 3.5. □

Since Key Lemma 3.16 contains an additional assumption, Theorem 2.12 cannot be applied immediately, and the proof of the FTAP in this model requires a bit of additional work.

Corollary 3.17. *The model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$ *satisfies the NGA condition if and only if there exists an equivalent uniformly integrable martingale measure (i.e.* $\mathcal{M} \neq \emptyset$*).*

Proof. *Step 1.* Let us prove the "only if" implication. Lemma 2.13 applied to the $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone $A_4(0)$ yields a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E_Q} X \le 0$ for any $X \in A$ that is bounded below. For any $i = 1, \dots, d$, any $u \le v \in \mathbb{R}_+$, and any $D \in \mathcal{F}_u$ such that $\overline{S}^i_u$ is bounded on D, the random variable $I_D(\overline{S}^i_v - \overline{S}^i_u)$ is bounded below, and hence, $\mathsf{E_Q} I_D(\overline{S}^i_v - \overline{S}^i_u) \le 0$. This shows that $\overline{S}^i$ is an $(\mathcal{F}_t, \mathsf{Q})$-supermartingale. By Doob's supermartingale convergence theorem, (see [RY99; Ch. II, Th. 2.10]), the limit $\lim_{t\to\infty} \overline{S}^i_t$ exists Q-a.s. and hence, P-a.s. Now, Theorem 2.12, combined with Key Lemma 3.16, yields the desired statement.

Step 2. Let us prove the "if" implication. Take $\mathsf{Q} \in \mathcal{M}$. Then by Doob's theorem, $\lim_{t\to\infty} \overline{S}_t$ exists Q-a.s. and hence, P-a.s. Now, Theorem 2.12, combined with Key Lemma 3.16, yields the desired statement. □

It has been shown in the proof of Corollary 3.17 that the NGA condition implies the existence of $\lim_{t\to\infty} \overline{S}_t$ P-a.s. Hence, Theorems 2.19 and 2.25 can be applied with no additional assumptions.

As an alternative to (3.12), one can define the set of attainable incomes in the model under consideration as

$$A = \Big\{ \sum_{n=1}^{N} \sum_{i=1}^{d} H^i_n (\overline{S}^i_{v_n} - \overline{S}^i_{u_n}) : N \in \mathbb{N},\ u_n \le v_n \text{ are } (\mathcal{F}_t)\text{-stopping times such that } v_n < \infty \ \mathsf{P}\text{-a.s., } H^i_n \text{ is } \mathcal{F}_{u_n}\text{-measurable} \Big\}.$$

For this choice of A, we can prove that $\mathcal{R} = \mathcal{M}$. However, we cannot prove that in this case Assumption 2.11 is satisfied.

Lemma 3.18. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$*, we have*

$$\mathcal{R} = \mathcal{M}.$$

Proof. *Step 1.* The inclusion $\mathcal{M} \subseteq \mathcal{R}$ follows from the similar inclusion in Key Lemma 3.16.

Step 2. Let us prove the inclusion $\mathcal{R} \subseteq \mathcal{M}$. Choose $\mathsf{Q} \in \mathcal{R}$. Fix $i \in \{1, \dots, d\}$. For any $u \le v \in \mathbb{R}_+$ and $D \in \mathcal{F}_u$, we have, due to the positivity of $\overline{S}^i$, that $\mathsf{E_Q} I_D(\overline{S}^i_v - \overline{S}^i_u) = 0$. Hence, $\overline{S}^i$ is an $(\mathcal{F}_t, \mathsf{Q})$-martingale.

By Doob's supermartingale convergence theorem, there exists a limit $\overline{S}^i_\infty =$ (a.s.) $\lim_{t\to\infty} \overline{S}^i_t$. By the Fatou lemma for conditional expectations,

$$\mathsf{E_Q}(\overline{S}^i_\infty \mid \mathcal{F}_t) \le \overline{S}^i_t, \quad t \ge 0. \tag{3.13}$$

In particular, $\mathsf{E}_\mathsf{Q}\overline{S}^i_\infty \le \mathsf{E}_\mathsf{Q}\overline{S}^i_0$.

Suppose that $\mathsf{E}_\mathsf{Q}\overline{S}^i_\infty < \mathsf{E}_\mathsf{Q}\overline{S}^i_0$. The process $X_t = \mathsf{E}_\mathsf{Q}(\overline{S}^i_\infty \mid \mathcal{F}_t)$, $t \ge 0$ has a càdlàg modification. Furthermore, $X_t \xrightarrow[t\to\infty]{\mathsf{Q}\text{-a.s.}} \overline{S}^i_\infty$. Consequently, the stopping time

$$\tau = \inf\left\{t \ge 0 : |\overline{S}^i_t - X_t| \le \frac{\mathsf{E}_\mathsf{Q}\overline{S}^i_0 - \mathsf{E}_\mathsf{Q}\overline{S}^i_\infty}{2}\right\}$$

is finite Q-a.s. It follows from the inclusion $\mathsf{Q} \in \mathcal{R}$ and the positivity of $\overline{S}^i$ that $\mathsf{E}_\mathsf{Q}\overline{S}^i_\tau = \mathsf{E}_\mathsf{Q}\overline{S}^i_0$. Thus,

$$\mathsf{E}_\mathsf{Q}\overline{X}_\tau > \mathsf{E}_\mathsf{Q}\overline{S}^i_0 - \frac{\mathsf{E}_\mathsf{Q}\overline{S}^i_0 - \mathsf{E}_\mathsf{Q}\overline{S}^i_\infty}{2} > \mathsf{E}_\mathsf{Q}\overline{S}^i_\infty.$$

But this contradicts the equality $\mathsf{E}_\mathsf{Q}X_\tau = \mathsf{E}_\mathsf{Q}\overline{S}^i_\infty$, which is a consequence of the optional stopping theorem for uniformly integrable martingales (see [RY99; Ch. II, Th. 3.2]). As a result, $\mathsf{E}_\mathsf{Q}\overline{S}^i_\infty = \mathsf{E}_\mathsf{Q}\overline{S}^i_0$. This, combined with (3.13), yields $\mathsf{E}_\mathsf{Q}(\overline{S}^i_\infty \mid \mathcal{F}_t) = \overline{S}^i_t$, $t \ge 0$. The proof is completed. □

The traditional approach to the arbitrage pricing in continuous-time models with the infinite time horizon is the same as the one for continuous-time models with a finite time horizon. The only difference is that the set of attainable incomes given by (3.4) should be replaced by

$$A = \left\{\int_0^\infty H_u d\overline{S}_u : H \text{ is } (\mathcal{F}_t)\text{-predictable, } \overline{S}\text{-integrable,} \right.$$
$$\left. \text{admissible, and such that } \lim_{t\to\infty} \int_0^t H_u d\overline{S}_u \text{ exists } \mathsf{P}\text{-a.s.}\right\}.$$

Here $\int_0^\infty H_u d\overline{S}_u := \lim_{t\to\infty} \int_0^t H_u d\overline{S}_u$. (This might be called the improper stochastic integral. Alternatively, one can use the stochastic integral up to infinity $\int_0^\infty H_u d\overline{S}_u$; see [CS04]. The FTAP remains the same for these two types of integrals.)

Many models with the infinite time horizon that are arbitrage-free in the traditional approach (i.e. satisfy the NFLVR condition for predictable admissible strategies) are not arbitrage-free in the proposed approach (i.e. do not satisfy the NGA condition for simple strategies). This is illustrated by the following example.

Example 3.19. Let $\overline{S}_t = e^{B_t - t/2}$, $t \in \mathbb{R}_+$, where B is a Brownian motion. Let $\mathcal{F}_t = \mathcal{F}^{\overline{S}}_t$, $\mathcal{F} = \bigvee_{t\ge 0} \mathcal{F}_t$. This model satisfies the NFLVR condition since the process $\overline{S}$ is a martingale (and hence, a sigma-martingale) with respect to the original probability measure. On the other hand, this model does not satisfy the NGA condition. Indeed, consider the stopping time $v = \inf\{t \in \mathbb{R}_+ : \overline{S}_t = 1/2\}$. Then the random variable $-\overline{S}_v + \overline{S}_0 = 1/2$ belongs to the set A given by (3.12). Hence, the NGA condition is violated.

From the financial point of view, the strategy providing the generalized arbitrage in this model consists in the short selling of the asset at time 0 and buying it back at time v. Note that this strategy is prohibited in the traditional approach by the admissibility condition. □

Remark. A *"buy and hold" strategy* consists in buying an asset, waiting until its discounted price reaches some higher level, and selling it back at that time. The opposite (it may be called "sell and wait") strategy consists in the short selling of an asset, waiting until its discounted price reaches some lower level and buying it back at that time. In many models (like the one described above) such "sell and wait" strategies lead to arbitrage opportunities. In the traditional approach, these strategies are prohibited by the admissibility condition. In the approach proposed here, such strategies are allowed, but the models, in which they yield arbitrage opportunities, are "prohibited" in the sense that they do not satisfy the NGA condition. Indeed, if the NGA condition is satisfied, then there exists an equivalent uniformly integrable martingale measure. But a uniformly integrable martingale with a strictly positive probability never reaches a preassigned level, so that in models satisfying the NGA condition the "sell and wait" strategy does not yield an arbitrage opportunity.

Remark. (Change of numéraire). Consider the problem of the change of numéraire. The setting of the problem is the same as in the previous section, but the time horizon is now infinite.

The same arguments as those used for the model with a finite time horizon (with the obvious changes) show that if $\lim_{t\to\infty}\overline{S}^1_t > 0$ P-a.s. (this limit exists P-a.s. if the model $(\Omega,\mathcal{F},\mathsf{P},A)$ satisfies the NGA condition), then the NGA property is preserved under the replacement of $\overline{S}$ by $\widetilde{S}$.

But without this additional assumption, the NGA property might not be preserved. As an example, consider the model with $S^0=1$ and $S^1_t = e^{B_{t\wedge\tau}-t\wedge\tau/2}$, where B is a Brownian motion and $\tau=\inf\{t\ge 0 : e^{B_s-s/2}\ge 2\}$. Let $\mathcal{F}_t=\mathcal{F}^S_t$, $\mathcal{F}=\bigvee_{t\ge 0}\mathcal{F}_t$. If we take the 0-th asset as a numériare, then, clearly, the NGA condition is satisfied. However, if we take the 1-st asset as a numéraire, then the NGA condition is not satisfied because $\lim_{t\to\infty}\widetilde{S}^0_t$ equals $+\infty$ with a strictly positive probability.

To conclude the section, we show that no "stationary" model with the infinite time horizon satisfies the NGA condition. We say that a process $(Z_t)_{t\in\mathbb{R}_+}$ has *stationary increments* if $Z_{t+h}-Z_{s+h}\overset{\text{Law}}{=}Z_t-Z_s$ for any $s\le t\in\mathbb{R}_+$, $h\in\mathbb{R}_+$.

Proposition 3.20. *Let $\overline{S}^i_t=\overline{S}_0 e^{Z^i_t}$, $i=1,\dots,d$, where Z has stationary increments and $\mathsf{P}(Z_t\ne Z_0)>0$ for some $t\in\mathbb{R}_+$. Then the NGA condition is not satisfied.*

Proof. Suppose that the NGA condition is satisfied. Without loss of generality, we can assume that $\mathsf{P}(Z^1_t\ne Z^1_0)>0$ for some $t\in\mathbb{R}_+$. The reasoning used in the proof of Corollary 3.17 shows that there exists $\lim_{t\to\infty}\overline{S}^1_t$ P-a.s. Hence, there exists $\lim_{t\to\infty}Z^1_t =: Z^1_\infty$ P-a.s. (this limit takes on values in $[-\infty,\infty)$). Denote $\mathsf{P}(Z^1_\infty>-\infty)$ by p. Fix $\varepsilon>0$ and find $N\in\mathbb{N}$ such that $N>1/\varepsilon$ and

$$\mathsf{P}(Z^1_\infty>-\infty \text{ and } |Z^1_n-Z^1_\infty|<\varepsilon \text{ for any } n\ge N)>p-\varepsilon.$$

Then

$$\mathsf{P}(Z^1_\infty>-\infty \text{ and } |Z^1_{2N}-Z^1_N|<2\varepsilon)>p-\varepsilon.$$

Since $Z^1_{2N}-Z^1_N\overset{\text{Law}}{=}Z^1_N$, we get $\mathsf{P}(|Z^1_N|<2\varepsilon)>p-\varepsilon$. As ε can be chosen arbitrarily small, we conclude that $\mathsf{P}(Z^1_\infty=0)=p$. Hence, $Z^1_\infty=0$ P-a.e. on the set $\{Z^1_\infty>-\infty\}$. This means that Z^1_∞ takes on only values $-\infty$ and 0.

Take $t \in \mathbb{R}_+$ such that $\mathsf{P}(Z_t^1 \neq Z_0^1) > 0$. Choose $\alpha \in \mathbb{R}_{++}$ such that $\mathsf{P}(|Z_t^1 - Z_0^1| > \alpha) > 0$. For any $T \in \mathbb{R}_+$,

$$\mathsf{P}(|Z_{T+t}^1 - Z_T^1| > \alpha) = \mathsf{P}(|Z_t^1 - Z_0^1| > \alpha) > 0.$$

Consequently, $\mathsf{P}(Z_\infty^1 = 0) < 1$. Thus, $\overline{S}_\infty^1$ takes on only values 0 and $\overline{S}_0^1$, and $\mathsf{P}(\overline{S}_\infty^1 = 0) > 0$. Then $\mathcal{M} = \emptyset$, and, by Corollary 3.17, the NGA condition is not satisfied. □

Corollary 3.21. *Let $\overline{S}_t^i = \overline{S}_0 e^{Z_t^i}$, $i = 1, \dots, d$, where Z is a Lévy process that is not identically equal to zero. Then the NGA condition is not satisfied.*

3.5 Model of the Term Structure of Interest Rates

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $\mathbb{T} \subseteq \mathbb{R}_+$ be the set of maturities of risk-free zero-coupon bonds. For any $T \in \mathbb{T}$, let $(S(t,T))_{t\in[0,T]}$ be a $[0,1]$-valued $(\mathcal{F}_t)$-adapted càdlàg process such that $S(T,T) = 1$. From the financial point of view, $S(t,T)$ is the price at time t of a risk-free zero-coupon bond with maturity T and face value 1. Let $(r_t)_{t\in\mathbb{R}_+}$ be an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted càdlàg process such that $\int_0^t r_s ds < \infty$ P-a.s for any $t \geq 0$. From the financial point of view, r_t is the instantaneous risk-free interest rate at time t (this is in contrast with the previous sections, where r is assumed to be constant). Set

$$\overline{S}(t,T) = \exp\biggl\{-\int_0^t r_s ds\biggr\} S(t,T), \quad T \in \mathbb{T},\ t \in [0,T].$$

Define the set of attainable incomes by

$$A = \biggl\{\sum_{n=1}^N H_n(\overline{S}(v_n,T_n) - \overline{S}(u_n,T_n)) : N \in \mathbb{N},\ T_n \in \mathbb{T},\ u_n \leq v_n \text{ are } [0,T_n]\text{-valued } (\mathcal{F}_t)\text{-stopping times, } H_n \text{ is } \mathcal{F}_{u_n}\text{-measurable}\biggr\}.$$

From the financial point of view, A is the set of incomes discounted to time 0 that can be obtained by trading risk-free zero-coupon bonds with various maturities (and using the bank account with the instantaneous risk-free interest rate to borrow/lend money).

`Notation`. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \text{for any } T \in \mathbb{T},\ (\overline{S}(t,T))_{t\in[0,T]} \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\}.$$

Key Lemma 3.22. *For the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}(0) = \mathcal{M}.$$

`Proof`. The statement is verified in the same way as Key Lemma 3.5. The proof here is even simpler since $\overline{S}(t,T)$ takes on values in $[0,1]$. □

3.6 Model with European Options

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $T \in \mathbb{R}_{++}$. Let S_T be an $\mathbb{R}_+^d$-valued random vector on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, S_T^i is the price of the i-th asset at time T (assets $1, \dots, d$ are the same as in Section 2.1). For $i = 1, \dots, d$, let $\mathbb{K}^i \subseteq \mathbb{R}_+$ be the set of strike prices K of traded *European call options* on the i-th asset with maturity T (the option holder receives the amount $(S_T^i - K)^+$ at time T). In practice $\mathbb{K}^i$ is finite, but in theory it is often assumed that $\mathbb{K}^i = \mathbb{R}_+$. Let $\varphi^i(K)$, $i = 1, \dots, d$, $K \in \mathbb{K}^i$ be the price at time 0 of a European call option on the i-th asset with maturity T and strike price K. Let $r \in \mathbb{R}_+$ be the risk-free interest rate. Define the set of attainable incomes by

$$A = \left\{ \sum_{n=1}^{N} \sum_{i=1}^{d} h_n^i \left(\frac{(S_T^i - K_n^i)^+}{1+r} - \varphi^i(K_n^i) \right) : N \in \mathbb{N},\ K_n^i \in \mathbb{K}^i,\ h_n^i \in \mathbb{R} \right\}.$$

From the financial point of view, A is the set of incomes discounted to time 0 that can be obtained by trading at times 0 and T European call options on assets $1, \dots, d$ with maturity T (and using the bank account to borrow/lend money). The possibility to trade the underlying assets at times 0 and T is easily incorporated into the model by assuming that $0 \in \mathbb{K}^i$, $i = 1, \dots, d$.

`Notation`. Set

$$\mathcal{D}^i = \Big\{ \mu : \mu \text{ is a probability measure on } \mathcal{B}(\mathbb{R}_+) \text{ such that } \int_{\mathbb{R}_+} (x - K)^+ \mu(dx) = (1+r)\varphi^i(K),\ K \in \mathbb{K}^i \Big\}, \quad i = 1, \dots, d.$$

Lemma 3.23. **(i)** *Let μ be a probability measure on $\mathcal{B}(\mathbb{R}_+)$ such that $\int_{\mathbb{R}_+} x\mu(dx) < \infty$. Then the function*

$$\varphi(K) = \int_{\mathbb{R}_+} (x - K)^+ \mu(dx), \quad K \in \mathbb{R}_+$$

satisfies the conditions:

(a) *φ is convex on $\mathbb{R}_+$;*
(b) $\varphi'_+(0) \ge -1$;
(c) $\lim_{K\to\infty} \varphi(K) = 0$.

Moreover, $\mu = \varphi''$, where φ'' is the second derivative of φ taken in the sense of distributions (i.e. $\varphi''((a,b]) = \varphi'_+(b) - \varphi'_+(a)$) with the convention: $\varphi''(\{0\}) = \varphi'_+(0) + 1$. (In what follows, we will always use this convention.)

(ii) *Suppose that φ satisfies conditions* (a)–(c). *Then the measure $\mu = \varphi''$ is a probability measure on $\mathcal{B}(\mathbb{R}_+)$ and*

$$\varphi(K) = \int_{\mathbb{R}_+} (x - K)^+ \mu(dx), \quad K \in \mathbb{R}_+.$$

`Proof`. *Step 1.* Let us prove (i). Conditions (a)–(c) are clear. Furthermore, $\varphi'_+(K) = -\mu((K, \infty))$, $K \in \mathbb{R}_+$, which implies that $\varphi'' = \mu$.

Step 2. Let us prove (ii). We have

$$\mu((a,b]) = \varphi'_+(b) - \varphi'_+(a), \quad a \le b \in \mathbb{R}_+.$$

It follows from condition (c) that $\lim_{K\to\infty}\varphi'_+(K)=0$, and hence, $\mu((a,\infty))=-\varphi'_+(a)$. Clearly, μ is a probability measure. For the function

$$\psi(K)=\int_{\mathbb{R}_+}(x-K)^+\mu(dx),\quad K\in\mathbb{R}_+,$$

we have

$$\psi'_+(K)=-\mu((K,\infty))=\varphi'_+(K),\quad K\in\mathbb{R}_+.$$

Since

$$\lim_{K\to\infty}\psi(K)=\lim_{K\to\infty}\varphi(K)=0,$$

we get $\varphi=\psi$. □

`Important remark`. By Lemma 3.23, an equivalent description of $\mathcal{D}^i$ is as follows:

$$\mathcal{D}^i=\Big\{\varphi'' : \varphi \text{ is convex on } \mathbb{R}_+,\ \varphi'_+(0)\ge -1,\ \lim_{K\to\infty}\varphi(K)=0,$$
$$\text{and } \varphi(K)=(1+r)\varphi^i(K),\ K\in\mathbb{K}^i\Big\}.$$

In particular, if $\mathbb{K}^i=\mathbb{R}_+$ and φ^i satisfies conditions (a)–(c) of Lemma 3.23, then $\mathcal{D}^i=\{(\varphi^i)''\}$. If $\mathbb{K}^i=\mathbb{R}_+$ and φ^i does not satisfy any of conditions (a)–(c) of Lemma 3.23, then $\mathcal{D}^i=\emptyset$.

`Notation`. Set $\mathcal{M}=\{\mathsf{Q}\sim\mathsf{P} : \mathrm{Law}_{\mathsf{Q}}\,S^i_T\in\mathcal{D}^i,\ i=1,\dots,d\}$.

Key Lemma 3.24. *Suppose that* $0\in\mathbb{K}^i$, $i=1,\dots,d$. *Then, for the model* $(\Omega,\mathcal{F},\mathsf{P},A)$, *we have*

$$\mathcal{R}=\mathcal{R}\Big(\sum_{i=1}^d(S^i_T-(1+r)\varphi^i(0))\Big)=\mathcal{M}.$$

This statement is clear (take Lemma 2.10 into account).

`Remarks`. `(i)` The condition $0\in\mathbb{K}^i$, $i=1,\dots,d$ means the possibility to trade the underlying assets at times 0 and T. So, from the financial point of view, this is not at all a restriction.

`(ii)` Let $d=1$ and $\mathbb{K}=\mathbb{R}_+$. It follows from Theorem 2.12 and Key Lemma 3.24 that the NGA condition is satisfied if and only if φ satisfies conditions (a)–(c) of Lemma 3.23 and the condition

(d) $(1+r)\varphi''\sim\mathrm{Law}_{\mathsf{P}}\,S_T$.

`(iii)` Let $i\in\{1,\dots,d\}$ and $\mathbb{K}^i=\mathbb{R}_+$. Suppose that the NGA condition is satisfied. Key Lemma 3.24 shows that $\mathrm{Law}_{\mathsf{Q}}\,S^i_T$ is the same for all $\mathsf{Q}\in\mathcal{R}$. This might be interpreted as follows: the market-estimated distribution of S^i_T is determined uniquely by the prices of European call options on the i-th asset with maturity T and all positive strike prices.

`(iv)` Let $i\in\{1,\dots,d\}$ and $\mathbb{K}^i=\mathbb{R}_+$. Suppose that the NGA condition is satisfied. Consider $F=f(S^i_T)$, where f is bounded below. It follows from Theorem 2.19 and Key Lemma 3.24 that

$$I(F)=\begin{cases}\Big\{\int_{\mathbb{R}_+}f(x)(1+r)(\varphi^i)''(dx)\Big\} & \text{if } \int_{\mathbb{R}_+}f(x)(\varphi^i)''(dx)<\infty,\\ \emptyset & \text{otherwise.}\end{cases}$$

In particular, the price of a *binary option* on the i-th asset (this is a contingent claim with the payoff $I(S_T^i \ge K)$ or $S_T^i I(S_T^i \ge K)$) is uniquely determined by the prices of European call options on the i-th asset with maturity T and all positive strike prices.

We conclude this section by three interesting examples. The first example shows that the ordinary NA condition (which means that $A \cap L_+^0 = \{0\}$) is too weak for the model under consideration.

Example 3.25. Let $d = 1$, $\mathbb{K} = \mathbb{R}_+$,

$$\mathsf{P}(S_T \in A) = \frac{1}{2}\Big(I(1 \in A) + \int_A e^{-x} dx\Big), \quad A \in \mathcal{B}(\mathbb{R}_+),$$

$\varphi(K) = e^{-K}$, and $r = 0$. This model satisfies the NA condition. Indeed, suppose that there exists

$$X = \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) \in A$$

such that $X \ge 0$ P-a.s. and $\mathsf{P}(X > 0) > 0$. Note that X can be represented as $X = f(S_T)$ with a continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}_+} f(x) e^{-x} dx = 0. \tag{3.14}$$

The above assumptions on X imply that $f \ge 0$ everywhere and f is not identically equal to zero. But this contradicts (3.14). Thus, NA is satisfied.

Consider now $F = I(S_T = 1)$. For every $\varepsilon > 0$, consider the function $f_\varepsilon(x) = (1 - \varepsilon^{-1}|x - 1|)^+$. Then the random variables

$$X_\varepsilon = f_\varepsilon(S_T) - \int_{\mathbb{R}_+} f_\varepsilon(x) e^{-x} dx$$

belong to A and

$$X_\varepsilon + \int_{\mathbb{R}_+} f_\varepsilon(x) e^{-x} dx \ge F \quad \mathsf{P}\text{-a.s.}$$

As $\int_{\mathbb{R}_+} f_\varepsilon(x) e^{-x} dx \underset{\varepsilon \downarrow 0}{\longrightarrow} 0$, it is reasonable to conclude that the fair price of F should not exceed 0 (thus, the fair price should equal 0 since F is positive). But on the other hand, $\mathsf{P}(F = 1) = 1/2$, so that we obtain a contradiction with the common sense. The reason is that this model is not "fair" because one can construct "asymptotic arbitrage" taking X_ε with $\varepsilon \downarrow 0$. □

The second example shows that the NFL condition (see Remark (iii) following Definition 2.8) is also too weak for the model under consideration.

Example 3.26. Let $d = 1$, $\mathbb{K} = \mathbb{R}_+$, $\mathsf{P}(S_T \le x) = 1 - e^{-x}$, $\varphi(K) = e^{-K} + 1$, and $r = 0$. This model satisfies the NFL condition. Indeed, let

$$X = \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) \in A$$

be bounded below. Note that X can be represented as $X = f(S_T)$ with a continous function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\lim_{x\to\infty} \frac{f(x)}{x} = \sum_{n=1}^{N} h_n.$$

The assumption on X implies that $\sum_{n=1}^{N} h_n \geq 0$. Then we can write

$$X \geq \sum_{n=1}^{N} h_n\big((S_T - K_n)^+ - e^{-K_n}\big) = g(S_T) - \int_{\mathbb{R}_+} g(x)e^{-x}dx = g(S_T) - \mathsf{E}_\mathsf{P} g(S_T)$$

with $g(x) = \sum_{n=1}^{N} h_n(x - K_n)^+$. This implies that for any $X \in A_4(0)$ ($A_4(0)$ is defined by (2.7)), we have $\mathsf{E}_\mathsf{P} X \leq 0$, so that NFL is satisfied.

On the other hand, in this model the price of a European call option tends to 1 as the strike price tends to $+\infty$, which contradicts the common sense. Thus, this model is not "fair" since one can construct "asymptotic arbitrage" by selling European call options with the strike price $K \to +\infty$. □

The third example shows that $V_*(F)$ and $V^*(F)$ might not coincide with the values $C_*(F)$ and $C^*(F)$ given by (2.13) and (2.14). Thus, in general the proposed approach to arbitrage pricing yields a finer interval of fair prices than the traditional approach based on sub- and superreplication.

Example 3.27. Let $d = 1$, $\mathbb{K} = \mathbb{R}_+$, $\mathsf{P}(S_T \leq x) = 1 - e^{-x}$, $\varphi(K) = e^{-K}$, and $r = 0$. This model satisfies the NGA condition since $\mathsf{P} \in \mathcal{M}$. Choose $D \in \mathcal{B}(\mathbb{R}_+)$ such that, for any $a < b \in \mathbb{R}_+$, the sets $D \cap [a, b]$ and $[a, b] \setminus D$ have a strictly positive Lebesgue measure. Consider $F = I(S_T \in D)$.

Let us find $C^*(F)$ defined by (2.14). Let $x \in \mathbb{R}$ and

$$X = \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) \in A$$

be such that $x + X \geq F$ P-a.s. Set

$$g(y) = \sum_{n=1}^{N} h_n(y - K_n)^+, \quad y \in \mathbb{R}_+.$$

We have

$$\begin{aligned} X &= g(S_T) - \sum_{n=1}^{N} h_n e^{-K_n} \\ &= g(S_T) - \sum_{n=1}^{N} h_n \int_{\mathbb{R}_+} (y - K_n)^+ e^{-y} dy \\ &= g(S_T) - \int_{\mathbb{R}_+} g(y)e^{-y}dy. \end{aligned}$$

Thus,

$$x + g(S_T) - \int_{\mathbb{R}_+} g(y)e^{-y}dy \geq I(S_T \in D) \quad \mathsf{P}\text{-a.s.}$$

Using the continuity of g and the properties of D, we get

$$x + g(z) - \int_{\mathbb{R}_+} g(y)e^{-y}dy \geq 1, \quad z \in \mathbb{R}_+.$$

This implies that $x \geq 1$. Consequently, $C^*(F) = 1$.

In a similar way one checks that $C_*(F) = 0$. On the other hand, it follows from Theorem 2.19 and Key Lemma 3.24 that

$$V_*(F) = V^*(F) = \int_D e^{-y}dy \in (0,1). \qquad \square$$

3.7 Model with Barrier Options

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $T \in \mathbb{R}_{++}$. Let S_T, M_T be $\mathbb{R}_+^d$-valued random vectors on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, S_T^i is the price of the i-th asset at time T (assets $1,\dots,d$ are the same as in Section 2.1) and M_T^i is the maximal price of the i-th asset on $[0,T]$. For $i = 1,\dots,d$, let $\mathbb{B}^i \subseteq \mathbb{R}_+^2$ be the set of pairs (K,L), for which there exists a traded *up-and-in call option* on the i-th asset with maturity T, strike price K, and barrier L (the option holder receives the amount $(S_T^i - K)^+ I(M_T^i \geq L)$ at time T). Let $\varphi^i(K,L)$, $i = 1,\dots,d$, $(K,L) \in \mathbb{B}^i$ be the price at time 0 of such an option. Let $r \in \mathbb{R}_+$ be the continuously compounded risk-free interest rate. Define the set of attainable incomes by

$$A = \biggl\{ \sum_{n=1}^N \sum_{i=1}^d h_n^i \bigl(e^{-rT}(S_T^i - K_n^i)^+ I(M_T^i \geq L_n^i) - \varphi^i(K_n^i, L_n^i)\bigr) : \\ N \in \mathbb{N},\ (K_n^i, L_n^i) \in \mathbb{B}^i,\ h_n^i \in \mathbb{R} \biggr\}.$$

The possibility to trade the underlying assets at times 0 and T is easily incorporated into the model by assuming that $(0,0) \in \mathbb{B}^i$, $i = 1,\dots,d$.

`Notation`. Set

$$\mathcal{D}^i = \biggl\{ \mu : \mu \text{ is a probability measure on } \mathcal{B}(\mathbb{R}_+^2) \text{ such that} \\ \int_{\mathbb{R}_+^2} (x-K)^+ I(y \geq L)\mu(dx,dy) = e^{rT}\varphi^i(K,L),\ (K,L) \in \mathbb{B}^i \biggr\}, \quad i = 1,\dots,d.$$

`Remark`. Let $\mathbb{B}^i = \mathbb{R}_+^2$. Then $\mathcal{D}^i$ is either a singleton or the empty set.

`Notation`. Set $\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \mathrm{Law}_{\mathsf{Q}}(S_T^i, M_T^i) \in \mathcal{D}^i,\ i = 1,\dots,d\}$.

Key Lemma 3.28. *Suppose that* $(0,0) \in \mathbb{K}^i$, $i = 1,\dots,d$. *Then, for the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}\biggl(\sum_{i=1}^d \bigl(S_T^i - e^{rT}\varphi^i(0,0)\bigr)\biggr) = \mathcal{M}.$$

This statement is clear (take Lemma 2.10 into account).

`Remarks.` `(i)` Let $i \in \{1, \dots, d\}$ and $\mathbb{B}^i = \mathbb{R}_+^2$. Suppose that the NGA condition is satisfied. Key Lemma 3.28 shows that $\mathrm{Law}_{\mathsf{Q}}(S_T^i, M_T^i)$ is the same for all $\mathsf{Q} \in \mathcal{R}$. This might be interpreted as follows: the market-estimated distribution of (S_T^i, M_T^i) is determined uniquely by the prices of up-and-in call options on the i-th asset with maturity T, all positive strike prices, and all positive barriers.

`(ii)` Let $i \in \{1, \dots, d\}$ and $\mathbb{B}^i = \mathbb{R}_+^2$. Suppose that the NGA condition is satisfied. Consider $F = f(S_T^i, M_T^i)$, where f is bounded below. It follows from Theorem 2.19 and Key Lemma 3.28 that

$$I(F) = \begin{cases} \left\{ \int_{\mathbb{R}_+^2} f(x,y) e^{rT} \mu^i(dx, dy) \right\} & \text{if } \int_{\mathbb{R}_+^2} f(x,y) \mu^i(dx,dy) < \infty, \\ \emptyset & \text{otherwise}, \end{cases}$$

where μ^i is the unique element of $\mathcal{D}^i$. In particular, the prices of an *up-and-in put option*, an *up-and-out call option*, and an *up-and-out put option* on the i-th asset (these are the contingent claims with the payoffs $(S_T^i - K)^- I(M_T^i \ge L)$, $(S_T^i - K)^+ I(M_T^i < L)$, and $(S_T^i - K)^- I(M_T^i < L)$) as well as the price of a *lookback option* on the i-th asset (it has the payoff $M_T^i - S_T^i$) are uniquely determined by the prices of up-and-in call options on the i-th asset with maturity T, all positive strike prices, and all positive barriers.

3.8 Model for Assessing Credit Risk

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Let ξ be a $[0, \infty]$-valued random variable on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, ξ is the time of default of some company. Let $\mathbb{T} \subseteq \mathbb{R}_+$ be the set of maturities of (risky) zero-coupon bonds issued by this company. Let $B(T)$, $T \in \mathbb{T}$ be the price at time 0 of a zero-coupon bond with maturity T and face value 1 issued by this company. Let $r(T)$, $T \in \mathbb{R}_+$ be the risk-free interest rate for the period $[0, T]$, i.e. $e^{-r(T)T}$ is the price at time 0 of a risk-free zero-coupon bond with maturity T and face value 1. Let us assume that there exists $\varkappa \in [0, 1)$ such that at time ξ each holder of a zero-coupon bond with maturity $T \ge \xi$ issued by the company receives the amount $\varkappa e^{-r(T)T + r(\xi)\xi}$ (when discounted to time 0, this amount equals $\varkappa e^{-r(T)T}$, which is $\varkappa$ times the discounted amount obtained by a holder of a risk-free zero-coupon bond with maturity T and face value 1). Define the set of attainable incomes by

$$A = \left\{ \sum_{n=1}^{N} h_n \bigl(e^{-r(T_n)T_n} (I(T_n < \xi) + \varkappa I(T_n \ge \xi)) - B(T_n) \bigr) : N \in \mathbb{N},\ T_n \in \mathbb{T},\ h_n \in \mathbb{R} \right\}.$$

`Notation.` Set

$$\mathcal{D} = \Bigl\{ \mu : \mu \text{ is a probability measure on } \mathcal{B}([0, \infty]) \text{ such that } \mu([0,T]) = \frac{1 - e^{r(T)T} B(T)}{1 - \varkappa},\ T \in \mathbb{T} \Bigr\},$$

$$\mathcal{M} = \{ \mathsf{Q} \sim \mathsf{P} : \mathrm{Law}_{\mathsf{Q}}\, \xi \in \mathcal{D} \}.$$

Key Lemma 3.29. *For the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}(0) = \mathcal{M}.$$

This statement is clear (take Lemma 2.10 into account).

Remark. Let $\mathbb{T} = \mathbb{R}_+$. Suppose that the NGA condition is satisfied. Key Lemma 3.29 shows that $\mathrm{Law}_{\mathsf{Q}}\, \xi$ is the same for all $\mathsf{Q} \in \mathcal{R}$. This might be interpreted as follows: the market-estimated distribution of ξ is determined uniquely (under the above assumptions) by the prices of zero-coupon bonds with all positive maturities issued by the company.

3.9 Mixed Model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_t)_{t\in[0,T]}$ be an $\mathbb{R}^d_+$-valued $(\mathcal{F}_t)$-adapted càdlàg process. From the financial point of view, S^i_t is the price of the i-th asset at time t (assets $1, \dots, d$ are the same as in Section 2.1). For $i = 1, \dots, d$, let $\mathbb{E}^i \subseteq [0, T] \times \mathbb{R}_+$ be the set of pairs (t, K), for which there exists a traded European call option on the i-th asset with maturity t and strike price K. Let $\varphi^i(t, K)$, $i = 1, \dots, d$, $(t, K) \in \mathbb{E}^i$ be the price at time 0 of such an option. Let $r \in \mathbb{R}_+$ be the risk-free interest rate. Define $\overline{S}$ by (3.2). Define the set of attainable incomes by

$$\begin{aligned} A = \Bigg\{ &\sum_{m=1}^{M} \sum_{i=1}^{d} H^i_m \big(\overline{S}^i_{v_m} - \overline{S}^i_{u_m}\big) + \sum_{n=1}^{N} \sum_{i=1}^{d} h^i_n \big(e^{-rt^i_n}\big(S^i_{t^i_n} - K^i_n\big)^+ - \varphi^i(t^i_n, K^i_n)\big) : \\ &M, N \in \mathbb{N},\ u_m \le v_m \text{ are } (\mathcal{F}_t)\text{-stopping times},\ H^i_m \text{ is } \mathcal{F}_{u_m}\text{-measurable}, \\ &(t^i_n, K^i_n) \in \mathbb{E}^i,\ h^i_n \in \mathbb{R} \Bigg\}. \end{aligned}$$

From the financial point of view, A is the set of incomes discounted to time 0 that can be obtained by trading assets $1, \dots, d$ on the interval $[0, T]$ and trading European call options on these assets at times 0 and T (as well as using the bank account to borrow/lend money).

Notation. Set

$$\begin{aligned} \mathbb{K}^i(t) &= \{K \in \mathbb{R}_+ : (t, K) \in \mathbb{E}^i\}, \quad i = 1, \dots, d,\ t \in [0, T], \\ \mathcal{D}^i(t) &= \Bigg\{ \mu : \mu \text{ is a probability measure on } \mathcal{B}(\mathbb{R}_+) \text{ such that} \\ &\qquad \int_{\mathbb{R}_+} (x - K)^+ \mu(dx) = e^{rt} \varphi^i(t, K),\ K \in \mathbb{K}^i(t) \Bigg\}, \quad i = 1, \dots, d,\ t \in [0, T], \end{aligned}$$

$$\begin{aligned} \mathcal{M} = \{&\mathsf{Q} \sim \mathsf{P} : \overline{S} \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale and} \\ &\mathrm{Law}_{\mathsf{Q}}\, S^i_t \in \mathcal{D}^i(t),\ i = 1, \dots, d,\ t \in [0, T]\}. \end{aligned}$$

Key Lemma 3.30. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}\bigg(\sum_{i=1}^{d} \big(\overline{S}^i_T - \overline{S}^i_0\big)\bigg) = \mathcal{M}.$$

Proof. Denote $\sum_{i=1}^{d} \big(\overline{S}^i_T - \overline{S}^i_0\big)$ by Z_0.
Step 1. The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 2.10.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(Z_0)$. The proof of Key Lemma 3.5 (Step 2) shows that $\overline{S}$ is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. Fix $i \in \{1, \dots, d\}$, $(t, K) \in \mathbb{E}^i$. There exists $\alpha \in \mathbb{R}_{++}$ such that $\alpha\overline{S}_t^i = S_t^i$. We have

$$\mathsf{E}_\mathsf{Q}\big(\alpha\overline{S}_t^i - \alpha\overline{S}_0^i - (S_t^i - K)^+ + e^{rt}\varphi^i(t,K)\big) = 0$$

since the random variable under the expectation belongs to A and is bounded. By the martingale property of $\overline{S}$, $\mathsf{E}_\mathsf{Q}\big(\alpha\overline{S}_t^i - \alpha\overline{S}_0^i\big) = 0$, which implies that

$$\mathsf{E}_\mathsf{Q}(S_t^i - K)^+ = e^{rt}\varphi^i(t,K).$$

As a result, $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Fix $X = X_1 + X_2 \in A$, where

$$X_1 = \sum_{m=1}^{M}\sum_{i=1}^{d} H_m^i\big(\overline{S}_{v_m}^i - \overline{S}_{u_m}^i\big),$$

$$X_2 = \sum_{n=1}^{N}\sum_{i=1}^{d} h_n^i\big(e^{-rt_n^i}\big(S_{t_n^i}^i - K_n^i\big)^+ - \varphi^i(t_n^i, K_n^i)\big).$$

Clearly, X_2 is Q-integrable and $\mathsf{E}_\mathsf{Q}X_2 = 0$. The proof of Key Lemma 3.5 (Step 3) shows that $\mathsf{E}_\mathsf{Q}X_1^- \ge \mathsf{E}_\mathsf{Q}X_1^+$. This leads to the inequality $\mathsf{E}_\mathsf{Q}X^- \ge \mathsf{E}_\mathsf{Q}X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

We conclude this section with a definition of the implied volatility that is alternative to the traditional one.

Definition 3.31. Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition. We call the values

$$\sigma_*^i(t) = \inf_{\mu \in \mathcal{D}^i}\bigg(\int_{\mathbb{R}_+} x^2\mu(dx) - \bigg(\int_{\mathbb{R}_+} x\mu(dx)\bigg)^2\bigg)^{1/2},$$

$$\sigma^{*i}(t) = \sup_{\mu \in \mathcal{D}^i}\bigg(\int_{\mathbb{R}_+} x^2\mu(dx) - \bigg(\int_{\mathbb{R}_+} x\mu(dx)\bigg)^2\bigg)^{1/2}$$

the *lower* and *upper implied volatility* of the i-th asset at time t. If $\sigma_*^i(t) = \sigma^{*i}(t)$, then we call this value the *implied volatility* of the i-th asset at time t.

4 Particular Models with Friction

Sections 4.1–4.4 are the "friction duals" of Sections 3.1, 3.3, 3.6, and 3.9.

We do not have the friction analog of the results of Section 3.4.

From the study below it is completely clear how to extend the models and the results of Sections 3.2, 3.5, 3.7, and 3.8 in order to incorporate friction. Therefore, we do not present explicitly the "friction duals" of these sections.

4.1 One-Period Model

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Let $S_0^a, S_0^b, S_0^c \in \mathbb{R}_+^d$ and S_1^a, S_1^b, S_1^c be $\mathbb{R}_+^d$-valued random vectors on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, $(S^a)_n^i$ is the *ask price* of the i-th asset at time n, i.e. the amount needed to buy this asset at this time; $(S^b)_n^i$ is the *bid price* of the i-th asset at time n, i.e. the amount obtained when one sells this asset at this time; $(S^c)_n^i$ is the amount obtained when one short sells the i-th asset at time n (so that $S^a \ge S^b \ge S^c$ componentwise). Define $\overline{S}^a$, $\overline{S}^b$, and $\overline{S}^c$ by (2.1). Define the set of attainable incomes by

$$A = \left\{ \sum_{i=1}^d \left[g^i \big((\overline{S}^b)_1^i - (\overline{S}^a)_0^i \big) + h^i \big(-(\overline{S}^a)_1^i + (\overline{S}^c)_0^i \big) \right] : g^i, h^i \in \mathbb{R}_+ \right\}.$$

`Remark`. The case $S^a = S^b = S^c$ corresponds to a frictionless model; the case $S^a = S^b$, $S^c = 0$ corresponds to a model with no transaction costs but with short sales prohibited; the case $S^c = S^b$ corresponds to a model with transaction costs an no costs of short selling, etc.

`Notation`. Set

$$\mathcal{M} = \left\{ \mathsf{Q} \sim \mathsf{P} : \mathsf{E_Q}(\overline{S}^b)_1^i \le (\overline{S}^a)_0^i \text{ and } \mathsf{E_Q}(\overline{S}^a)_1^i \ge (\overline{S}^c)_0^i, \ i = 1, \dots, d \right\}$$

(here $\mathsf{E_Q}(\overline{S}^a)_1^i$ may take on the value $+\infty$).

Key Lemma 4.1. *Suppose that for any* $i = 1, \dots, d$, *either* $(S^c)^i = 0$ *or there exists* $\gamma^i \in \mathbb{R}_{++}$ *such that* $(\overline{S}^a)^i \le \gamma^i (\overline{S}^b)^i$. *Then for the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}\bigg(\sum_{i=1}^d \big((\overline{S}^b)_1^i - (\overline{S}^a)_0^i \big) \bigg) = \mathcal{M}.$$

This statement is clear (take Lemma 2.10 into account).

The theorem below shows that in this model one can deal with the NA condition instead of the NGA condition.

Theorem 4.2 (FTAP). *Under the assumptions of Key Lemma 4.1, the following conditions are equivalent:*

(a) *NGA*;
(b) *NA* (*i.e.* $A \cap L^0_+ = \{0\}$);
(c) $\mathcal{M} \neq \emptyset$.

Proof. *Step 1.* The implication (a)⇒(b) is obvious.

Step 2. Let us prove the implication (b)⇒(c). Consider the measure $\mathsf{P}' = c(\|S^a_1\| \vee 1)^{-1}\mathsf{P}$, where c is the normalizing constant. The set A is a closed convex cone in $L^1(\mathsf{P}')$. By the "L^1 version" of the Kreps–Yan theorem (it is the same as Lemma 2.13 with L^∞ replaced by L^1 and $\sigma(L^\infty, L^1)$ replaced by the norm topology of L^1; the proof can be found in [Y80]), there exists a probability measure $\mathsf{Q} \sim \mathsf{P}'$ such that the density $d\mathsf{Q}/d\mathsf{P}'$ is bounded and $\mathsf{E}_\mathsf{Q} X \leq 0$ for any $X \in A$. Then $\mathsf{Q} \in \mathcal{M}$.

Step 3. The implication (c)⇒(a) follows from Theorem 2.12. □

Remarks. (i) Let $F \in L^0$ be bounded below. It follows from Theorem 4.2 that the objects $I(F)$, $V_*(F)$, and $V^*(F)$ would remain unchanged if we replaced the NGA condition in their definition by the NA condition.

(ii) Consider a model with proportional transaction costs and proportional costs of short selling, i.e. a model with

$$(S^a)^i_n = S^i_n, \quad (S^b)^i_n = (1-\alpha^i)S^i_n, \quad (S^c)^i_n = (1-\alpha^i-\beta^i)S^i_n,$$

where $S_0 \in \mathbb{R}^d_+$, S_1 is an $\mathbb{R}^d_+$-valued random vector, $\alpha^i \in [0,1)$, $\beta^i \in [0,1]$, $\alpha^i + \beta^i \leq 1$. Set $C = \overline{\operatorname{conv}} \operatorname{supp} \operatorname{Law}_\mathsf{P} \overline{S}_1$ and let C° denote the relative interior of C. Set

$$\begin{aligned} \widetilde{C^\circ} = \{\widetilde{x} \in \mathbb{R}^d_+ :\ & \text{there exists } x \in C^\circ \text{ such that} \\ & (1-\alpha^i)x^i \leq \widetilde{x}^i \leq (1-\alpha^i-\beta^i)^{-1}x^i,\ i = 1,\dots,d\} \end{aligned}$$

It is easy to see that conditions (a)–(c) of Theorem 4.2 are equivalent to

(d) $\overline{S}_0 \in \widetilde{C^\circ}$.

(iii) Consider a model with proportional transaction costs and proportional costs of short selling described above. Let $F \in L^0$ be bounded below. Set $D = \overline{\operatorname{conv}} \operatorname{supp} \operatorname{Law}_\mathsf{P}(F, \overline{S}_1)$ and let D° denote the relative interior of D. Set

$$\begin{aligned} \widetilde{D^\circ} = \{(\widetilde{x}, \widetilde{y}) \in \mathbb{R}_+ \times \mathbb{R}^d_+ : & \text{ there exists } (x,y) \in D^\circ \text{ such that} \\ & \widetilde{x} = x \text{ and } (1-\alpha^i)y^i \leq \widetilde{y}^i \leq (1-\alpha^i-\beta^i)^{-1}y^i,\ i = 1,\dots,d\} \end{aligned}$$

It follows from the previous remark that

$$I(F) = \big\{x : (x, \overline{S}_0) \in \widetilde{D^\circ}\big\}.$$

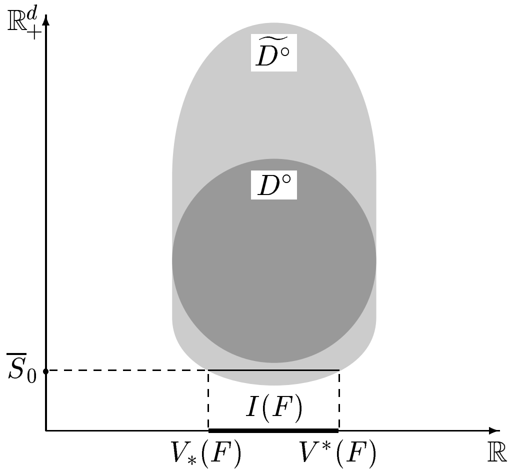


Figure 6. The joint arrangement of $I(F)$, $V_*(F)$, $V^*(F)$, D°, and $\widetilde{D^\circ}$ in a model with proportional transaction costs

4.2 Continuous-Time Model with a Finite Time Horizon

In this section we consider a model with friction on 3 levels of generality:

1. model with arbitrary transaction costs and arbitrary costs of short selling;
2. model with arbitrary transaction costs and no costs of short selling;
3. model with proportional transaction costs and proportional costs of short selling.

1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial, while $(\mathcal{F}_t)$ is right-continuous and complete. Let $(S^a_t)_{t\in[0,T]}$, $(S^b_t)_{t\in[0,T]}$, and $(S^c_t)_{t\in[0,T]}$ be $\mathbb{R}^d_+$-valued $(\mathcal{F}_t)$-adapted càdlàg processes. From the financial point of view, $(S^a)^i_t$ is the ask price of the i-th asset at time t, $(S^b)^i_t$ is the bid price of the i-th asset at time t, and $(S^c)^i_t$ is the amount one gets when short selling the i-th asset at time t. Define $\overline{S}^a$, $\overline{S}^b$, and $\overline{S}^c$ by (3.2). Define the set of attainable incomes by

$$A = \Bigg\{\sum_{n=1}^{N}\sum_{i=1}^{d}\big[G^i_n\big((\overline{S}^b)^i_{v_n} - (\overline{S}^a)^i_{u_n}\big) + H^i_n\big(-(\overline{S}^a)^i_{v_n} + (\overline{S}^c)^i_{u_n}\big)\big] : N \in \mathbb{N},$$
$$u_n \le v_n \text{ are } (\mathcal{F}_t)\text{-stopping times, } G^i_n, H^i_n \text{ are } \mathbb{R}_+\text{-valued and } \mathcal{F}_{u_n}\text{-measurable}\Bigg\}.$$

Notation. (i) Set

$$\mathcal{M} = \big\{\mathsf{Q} \sim \mathsf{P} : \text{for any } i = 1, \dots, d \text{ and any } (\mathcal{F}_t)\text{-stopping times } u \le v,$$
$$\mathsf{E}_\mathsf{Q}\big((\overline{S}^b)^i_v \mid \mathcal{F}_u\big) \le (\overline{S}^a)^i_u \text{ and } \mathsf{E}_\mathsf{Q}\big((\overline{S}^a)^i_v \mid \mathcal{F}_u\big) \ge (\overline{S}^c)^i_u\big\}.$$

(ii) Let $t_1, t_2, \dots$ be a numbering of $\{qT : q \in \mathbb{Q} \cap [0,1]\}$. Set

$$Z_0 = \sum_{n=1}^{\infty}\sum_{i=1}^{d} 2^{-n}\big((\overline{S}^b)^i_{t_n} - (\overline{S}^a)^i_0\big).$$

(This series converges P-a.s. since $\overline{S}^b$ is càdlàg.)

Key Lemma 4.3. *Suppose that for any $i = 1, \dots, d$, either $(\overline{S}^c)^i = 0$ or there exists $\gamma^i \in \mathbb{R}_{++}$ such that $(\overline{S}^a)^i \le \gamma^i (\overline{S}^c)^i$. Then for the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}(Z_0) = \mathcal{M}.$$

`Proof.` *Step 1.* The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 2.10.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(Z_0)$. Fix $i \in \{1, \dots, d\}$ and $(\mathcal{F}_t)$-stopping times $u \le v$. For $n \in \mathbb{N}$, set

$$u_n = \sum_{k=1}^{n} \frac{kT}{n} I\biggl(\frac{(k-1)T}{n} < u \le \frac{kT}{n}\biggr),$$
$$v_n = \sum_{k=1}^{n} \frac{kT}{n} I\biggl(\frac{(k-1)T}{n} < v \le \frac{kT}{n}\biggr).$$

Then for any $n \le m$ and any $D \in \mathcal{F}_{u_m}$ such that $(\overline{S}^a)^i_{u_m}$ is bounded on D, we have $u_m \le v_n$ and

$$\mathsf{E}_\mathsf{Q} I_D\bigl((\overline{S}^b)^i_{v_n} - (\overline{S}^a)^i_{u_m}\bigr) \le 0,$$

which implies that

$$\mathsf{E}_\mathsf{Q}\bigl((\overline{S}^b)^i_{v_n} \mid \mathcal{F}_{u_m}\bigr) \le (\overline{S}^a)^i_{u_m}. \tag{4.1}$$

As u_m decreases to u pointwise, we have $\mathcal{F}_{u_m} \subseteq \mathcal{F}_{u_{m-1}}$ and $\bigcap_{m=1}^\infty \mathcal{F}_{u_m} = \mathcal{F}_u$ (see [RY99; Ch. I, Ex. 4.17]). Therefore,

$$\mathsf{E}_\mathsf{Q}\bigl((\overline{S}^b)^i_{v_n} \,\big|\, \mathcal{F}_{u_m}\bigr) \xrightarrow[m\to\infty]{\mathsf{Q}\text{-a.s.}} \mathsf{E}_\mathsf{Q}\bigl((\overline{S}^b)^i_{v_n} \,\big|\, \mathcal{F}_u\bigr)$$

(see [RY99; Ch. II, Cor. 2.4]) and (4.1) yields

$$\mathsf{E}_\mathsf{Q}\bigl((\overline{S}^b)^i_{v_n} \mid \mathcal{F}_u\bigr) \le (\overline{S}^a)^i_u.$$

Applying the Fatou lemma for conditional expectations, we get

$$\mathsf{E}_\mathsf{Q}\bigl((\overline{S}^b)^i_v \mid \mathcal{F}_u\bigr) \le (\overline{S}^a)^i_u.$$

It remains to prove the inequality

$$\mathsf{E}_\mathsf{Q}\bigl((\overline{S}^a)^i_v \mid \mathcal{F}_u\bigr) \ge (\overline{S}^c)^i_u. \tag{4.2}$$

If $(\overline{S}^c)^i = 0$, this inequality is clear. Let us assume that $(\overline{S}^c)^i \ne 0$, which means that $(\overline{S}^a)^i \le \gamma^i (\overline{S}^c)^i$ with some $\gamma^i \in \mathbb{R}_{++}$. Then, for any $D \in \mathcal{F}_{u_m}$,

$$\mathsf{E}_\mathsf{Q} I_D\bigl(-(\overline{S}^a)^i_{v_n} + (\overline{S}^c)^i_{u_m}\bigr) \le 0$$

(we have used the inequality $(\overline{S}^a)^i \le \gamma^i ((\overline{S}^b)^i)$), which implies that

$$\mathsf{E}_\mathsf{Q}\bigl((\overline{S}^a)^i_{v_n} \mid \mathcal{F}_{u_m}\bigr) \ge (\overline{S}^c)^i_{u_m}.$$

Arguing similarly as above, we get

$$\mathsf{E}_\mathsf{Q}\bigl((\overline{S}^a)^i_{v_n} \mid \mathcal{F}_u\bigr) \ge (\overline{S}^c)^i_u. \tag{4.3}$$

It follows that for any $(\mathcal{F}_t)$-stopping time v,

$$(\overline{S}^a)^i_v \le \gamma^i (\overline{S}^c)^i_v \le \gamma^i \mathsf{E}_{\mathsf{Q}}\big((\overline{S}^a)^i_T \mid \mathcal{F}_v\big) \le (\gamma^i)^2 \mathsf{E}_{\mathsf{Q}}\big((\overline{S}^b)^i_T \mid \mathcal{F}_v\big).$$

Using the inclusion $\mathsf{Q} \in \mathcal{R}(Z_0)$, it is easy to check that $(\overline{S}^b)^i_T$ is Q-integrable, and hence, the collection $\big((\overline{S}^a)^i_{v_n}\big)_{n=1}^\infty$ is Q-uniformly integrable. Now (4.2) follows from (4.3).

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Fix

$$\xi = \sum_{n=1}^N \sum_{i=1}^d \big[G^i_n\big((\overline{S}^b)^i_{v_n} - (\overline{S}^a)^i_{u_n}\big) + H^i_n\big(-(\overline{S}^a)^i_{v_n} + (\overline{S}^c)^i_{u_n}\big)\big] \in A.$$

It follows from Lemma 4.4 that $\mathsf{E}_{\mathsf{Q}}\xi^- \ge \mathsf{E}_{\mathsf{Q}}\xi^+$. Hence, $\mathsf{Q} \in \mathcal{R}$. □

Lemma 4.4. *Let X, Y, Z be $\mathbb{R}^d_+$-valued $(\mathcal{F}_t)$-adapted càdlàg processes such that for any $(\mathcal{F}_t)$-stopping times $u \le v$ and any $i = \{1, \dots, d\}$, we have*

$$\mathsf{E}(Y^i_v \mid \mathcal{F}_u) \le X^i_u, \quad \mathsf{E}(X^i_v \mid \mathcal{F}_u) \ge Z^i_u.$$

Let $N \in \mathbb{N}$, $u_n \le v_n$ be $(\mathcal{F}_t)$-stopping times, G_n, H_n be $\mathbb{R}^d_+$-valued $\mathcal{F}_{u_n}$-measurable random vectors, $n = 1, \dots, N$. Let σ be an $(\mathcal{F}_t)$-stopping time such that $\sigma \le u_1 \wedge \dots \wedge u_N$. Then, for the random variable

$$\xi = \sum_{n=1}^N \sum_{i=1}^d \big[G^i_n\big(Y^i_{v_n} - X^i_{u_n}\big) + H^i_n\big(-X^i_{v_n} + Z^i_{u_n}\big)\big],$$

we have

$$\mathsf{E}(\xi^- \mid \mathcal{F}_\sigma) \ge \mathsf{E}(\xi^+ \mid \mathcal{F}_\sigma).$$

(Here $\mathsf{E}(\xi^- \mid \mathcal{F}_\sigma)$ and $\mathsf{E}(\xi^+ \mid \mathcal{F}_\sigma)$ can take on the value $+\infty$.)

Proof. We will prove this statement by the induction in N.

Base of induction. Let $N = 1$. Clearly, $\mathsf{E}(\xi \mid \mathcal{F}_{u_1}) < \infty$ a.s. and

$$\mathsf{E}(\xi^- \mid \mathcal{F}_{u_1}) \ge \mathsf{E}(\xi^+ \mid \mathcal{F}_{u_1}).$$

This yields the desired result.

Step of induction. Assume that the statement is true for $N - 1$. Let us prove it for N. Set $\tau = u_1 \wedge \dots \wedge u_N$,

$$D_1 = \{\tau = u_1\}, \quad D_2 = \{\tau = u_2\} \setminus D_1, \dots, D_N = \{\tau = u_N\} \setminus \bigcup_{n=1}^{N-1} D_n$$

(note that $D_1, \dots, D_N \in \mathcal{F}_\tau$). Fix $m \in \{1, \dots, N\}$ and set

$$\zeta = \sum_{i=1}^d \big[G^i_m\big(Y^i_{v_m} - X^i_{u_m}\big) + H^i_m\big(-X^i_{v_m} + Z^i_{u_m}\big)\big],$$

$$\eta = \sum_{n \ne m} \sum_{i=1}^d \big[G^i_n\big(Y^i_{v_n} - X^i_{u_n}\big) + H^i_n\big(-X^i_{v_n} + Z^i_{u_n}\big)\big] = \xi - \zeta.$$

Note that $\{C \cap D_m : C \in \mathcal{F}_\tau\} = \{C \cap D_m : C \in \mathcal{F}_{u_m}\}$. Using this property, we get

$$\mathsf{E}(I_{D_m}|\zeta| \,|\, \mathcal{F}_\tau) = \mathsf{E}(I_{D_m}|\zeta| \,|\, \mathcal{F}_{u_m}) < \infty \quad \text{a.s.}$$

and

$$\mathsf{E}(I_{D_m}\zeta^- \mid \mathcal{F}_\tau) = \mathsf{E}(I_{D_m}\zeta^- \mid \mathcal{F}_{u_m}) \geq \mathsf{E}(I_{D_m}\zeta^+ \mid \mathcal{F}_{u_m}) = \mathsf{E}(I_{D_m}\zeta^+ \mid \mathcal{F}_\tau).$$

It follows from the induction assumption that

$$\mathsf{E}(\eta^- \,|\, \mathcal{F}_\tau) \geq \mathsf{E}(\eta^+ \,|\, \mathcal{F}_\tau).$$

Consequently,

$$I_{D_m}\mathsf{E}(\xi^- \,|\, \mathcal{F}_\tau) \geq I_{D_m}\mathsf{E}(\xi^+ \,|\, \mathcal{F}_\tau).$$

As m has been chosen arbitrarily, we get

$$\mathsf{E}(\xi^- \,|\, \mathcal{F}_\tau) \geq \mathsf{E}(\xi^+ \,|\, \mathcal{F}_\tau),$$

which yields the desired statement. □

2. Let us now consider a model with no costs of short selling, i.e. a model with $S^c = S^b$. In order to establish the structure of risk-neutral measures in this model, we need several auxiliary statements. The first one has been proved by F. Delbaen and W. Schachermayer [DS94; Appendix].

Lemma 4.5. *Let $\xi_1, \xi_2, \dots$ be $\mathbb{R}_+$-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$. Then there exists a sequence $(\eta_n)_{n\in\mathbb{N}}$ such that $\eta_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$, $n \in \mathbb{N}$ and (η_n) converges a.s. to a $[0, +\infty]$-valued random variable* ("conv" *denotes the convex hull*).

The next result is of independent interest. It has been proved by Choulli and Stricker [CS98] under some additional assumption. Here we give a simpler proof and get rid of this assumption.

Theorem 4.6. *Let X be a supermartingale and Y be a submartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ with a right-continuous and complete filtration (X and Y are not necessarily càdlàg). Suppose that $X_t \leq Y_t$ a.s., $t \in [0, T]$. Then there exists an $(\mathcal{F}_t)$-martingale M such that $X_t \leq M_t \leq Y_t$ a.s., $t \in [0, T]$.*

`Proof.` *Step 1.* Let us first prove this statement in the discrete-time setting, i.e. for the processes $(X_n)_{n=0,\dots,N}$ and $(Y_n)_{n=0,\dots,N}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P})$. Construct $(M_n)_{n=0,\dots,N}$ going from 0 to N by the following procedure: we set $M_0 = X_0$ and if $M_0, \dots, M_n$ are already constructed, we find an $\mathcal{F}_n$-measurable random variable λ_n such that

$$M_n = \lambda_n \mathsf{E}(X_{n+1} \mid \mathcal{F}_n) + (1 - \lambda_n)\mathsf{E}(Y_{n+1} \mid \mathcal{F}_n)$$

and set

$$M_{n+1} = \lambda_n X_{n+1} + (1 - \lambda_n) Y_{n+1}.$$

Then M is an $(\mathcal{F}_n)$-martingale, $0 \leq \lambda_n \leq 1$, and $X_n \leq M_n \leq Y_n$ a.s., $n = 0, \dots, N$.

Step 2. Consider now the continuous-time case. There exists a countable set $\mathbb{T} \subseteq [0, T]$ such that $\mathbb{T}$ is dense in $[0, T]$ and the functions $t \mapsto \mathsf{E}X_t$ and $t \mapsto \mathsf{E}Y_t$ are continuous on $[0, T] \setminus \mathbb{T}$. Let $t_1, t_2, \dots$ be a numbering of $\mathbb{T}$. For each $n \in \mathbb{N}$, we can find a random sequence $(M_k^n)_{k=0,\dots,n}$ that is a martingale with respect to $(\mathcal{F}_{t_k})_{k=0,\dots,n}$

and is such that $X_{t_k} \le M_k^n \le Y_{t_k}$ a.s., $k = 0, \dots, n$. Applying Lemma 4.5 to the random variables $\xi_n = M_n^n$, we construct a sequence (η_n) such that $\eta_n \in \operatorname{conv}(\xi_n, \xi_{n+1}, \dots)$ and (η_n) converges a.s. to a random variable η. Notice that $X_T \le \eta_n \le Y_T$ a.s. ($\mathbb{T}$ can be chosen in such a way that $T \in \mathbb{T}$), so that $\eta_n \xrightarrow[n\to\infty]{L^1(\mathsf{P})} \eta$. Consider the process $M_t = \mathsf{E}(\eta \mid \mathcal{F}_t)$, $t \in [0, T]$ (we take its càdlàg version). Then the inequalities

$$X_{t_k} \le \mathsf{E}(\eta_m \mid \mathcal{F}_{t_k}) \le Y_{t_k}, \quad n \in \mathbb{N},\ m \ge n,\ k = 0, \dots, n$$

imply that $X_t \le M_t \le Y_t$ a.s. for $t \in \mathbb{T}$. The processes X and Y admit modifications that are right-continuous on $[0, T] \setminus \mathbb{T}$ (see [IW89; Ch. I, Th. 6.9]). As a result, $X_t \le M_t \le Y_t$ a.s. for any $t \in [0, T]$. □

The next lemma clarifies the structure of risk-neutral measures in the model under consideration.

Lemma 4.7. *Let X and Y be $(\mathcal{F}_t)$-adapted càdlàg processes such that for any $(\mathcal{F}_t)$-stopping time τ, the random variables X_τ and Y_τ are integrable. This pair of processes satisfies the condition*

$$\mathsf{E}(X_v \mid \mathcal{F}_u) \le Y_u, \quad \mathsf{E}(Y_v \mid \mathcal{F}_u) \ge X_u$$

for any $(\mathcal{F}_t)$-stopping times $u \le v$ if and only if there exists an $(\mathcal{F}_t)$-martingale M such that $X \le M \le Y$ a.s.

`Proof`. *Step 1.* Let us prove the "only if" implication. Set

$$\widetilde{X}_t = \operatorname*{esssup}_{\tau \in \mathfrak{M}_t} \mathsf{E}(X_\tau \mid \mathcal{F}_t), \quad t \in [0, T],$$
$$\widetilde{Y}_t = \operatorname*{essinf}_{\tau \in \mathfrak{M}_t} \mathsf{E}(Y_\tau \mid \mathcal{F}_t), \quad t \in [0, T],$$

where $\mathfrak{M}_t$ denotes the set of $(\mathcal{F}_t)$-stopping times τ such that $\tau \ge t$. (Here $\operatorname{esssup}_\tau \zeta_\tau$ means the essential supremum of a family of random variables, i.e. a random variable ζ such that for any τ, $\zeta \ge \zeta_\tau$ a.s. and for any other random variable ζ' with this property, we have $\zeta \le \zeta'$; $\operatorname{essinf}_\tau \zeta_\tau$ is defined similarly.) As $\mathfrak{M}_s \supseteq \mathfrak{M}_t$ for $s \le t$, the process $\widetilde{X}$ is an $(\mathcal{F}_t)$-supermartingale, while $\widetilde{Y}$ is an $(\mathcal{F}_t)$-submartingale.

Let us prove that $\widetilde{X}_t \le \widetilde{Y}_t$ a.s., $t \in [0, T]$. Assume that there exists t such that $\mathsf{P}(\widetilde{X}_t > \widetilde{Y}_t) > 0$. Then there exist $\tau, \sigma \in \mathfrak{M}_t$ such that

$$\mathsf{P}(\mathsf{E}(X_\tau \mid \mathcal{F}_t) > \mathsf{E}(Y_\sigma \mid \mathcal{F}_t)) > 0.$$

This implies that $\mathsf{P}(\xi > \eta) > 0$, where $\xi = \mathsf{E}(X_\tau \mid \mathcal{F}_{\tau\wedge\sigma})$ and $\eta = \mathsf{E}(Y_\sigma \mid \mathcal{F}_{\tau\wedge\sigma})$. Assume first that $\mathsf{P}(\{\xi > \eta\} \cap \{\tau \le \sigma\}) > 0$. On the set $\{\tau \le \sigma\}$, we have

$$\xi = X_\tau = X_{\tau\wedge\sigma},$$
$$\eta = \mathsf{E}(Y_\sigma \mid \mathcal{F}_{\tau\wedge\sigma}) = \mathsf{E}(Y_{\tau\vee\sigma} \mid \mathcal{F}_{\tau\wedge\sigma}),$$

which yields a contradiction. In a similar way, we arrive at a contradiction under the assumption $\mathsf{P}(\{\xi > \eta\} \cap \{\tau \ge \sigma\}) > 0$. As a result, $\widetilde{X} \le \widetilde{Y}$. Now, an application of Theorem 4.6 yields the existence of a desired process M.

Step 2. The "if" implication follows immediately from the optional stopping theorem (see [RY99; Ch. II, Th. 3.2]). □

`Notation`. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \text{for any } i = 1, \dots, d, \text{ there exists an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale } M^i \text{ such that } (\overline{S}^b)^i \le M^i \le (\overline{S}^a)^i\}.$$

Key Lemma 4.8. *Suppose that for any $i = 1, \dots, d$, there exists $\gamma^i \in \mathbb{R}_{++}$ such that $(\overline{S}^a)^i \le \gamma^i (\overline{S}^b)^i$. Then for the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}(Z_0) = \mathcal{M}.$$

`Proof`. This statement is a consequence of Key Lemma 4.3 and Lemma 4.7. □

`Remark`. As a corollary, the model satisfies the NGA condition if and only if $\mathcal{M} \neq \emptyset$. This agrees with the result of Jouini and Kallal [JK95]. However, there are several differences between our approach and the one in [JK95]. The most important one is that Jouini and Kallal work with the L^2-setting (in particular, the random variables $(S^a)^i_t$ are assumed to be square-integrable and the densities $d\mathsf{Q}/d\mathsf{P}$ or risk-neutral measures should also be square-integrable), while we work with the L^0-setting.

3. Finally, we consider a model with proportional transaction costs and proportional costs of short selling, i.e. a model with

$$(S^a)^i = S^i, \quad (S^b)^i = (1 - \alpha^i) S^i, \quad (S^c)^i = (1 - \alpha^i - \beta^i) S^i,$$

where S is an $\mathbb{R}^d_+$-valued $(\mathcal{F}_t)$-adapted càdlàg process, $\alpha^i \in [0, 1)$, $\beta^i \in [0, 1]$, $\alpha^i + \beta^i \le 1$.

Definition 4.9. Let $a, b \in [0, 1]$. An $\mathbb{R}_+$-valued process X is called an $(\mathcal{F}_t, \mathsf{P})$*-delta-martingale of order* (a, b) (or $(\mathcal{F}_t, \mathsf{P}, a, b)$*-delta-martingale*, or $(\mathcal{F}_t, \mathsf{P})$-$\Delta$-*martingale* of order (a, b)) if

(i) X is $(\mathcal{F}_t)$-adapted and càdlàg;
(ii) $\mathsf{E}_\mathsf{P} X_t < \infty$, $t \in [0, T]$;
(iii) for any $(\mathcal{F}_t)$-stopping times $u \le v$, we have

$$aX_u \le \mathsf{E}_\mathsf{P}(X_v \mid \mathcal{F}_u) \le b^{-1} X_u \tag{4.4}$$

(if $b = 0$, then the second inequality is omitted).

`Remarks`. `(i)` If $a = b = 1$, then the above definition is equivalent to the definition of a martingale. If $a = 0$, $b = 1$, then the above definition is equivalent to the definition of a supermartingale. If $a = 1$, $b = 0$, then the above definition is equivalent to the definition of a submartingale. These statements follow from the optional stopping theorem (see [RY99; Ch. II, Th. 3.3]).

`(ii)` Gnedenko [G04] introduced the notion of an *epsilon-martingale* in connection with some optimal control problem in models with transaction costs. The definition of an epsilon-martingale is similar to that of delta-martingale with (4.4) replaced by: $|\mathsf{E}_\mathsf{P}(X_v \mid \mathcal{F}_u) - X_u| \le \varepsilon$, where $\varepsilon \in \mathbb{R}_+$.

In the definition of a martingale, supermartingale, or submartingale, one can use deterministic times u, v (this leads to an equivalent definition). This is not the case for the delta-martingales as shown by the example below.

Example 4.10. Let $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}_2 = 2^\Omega$, $\mathsf{P}(\omega_1) = \mathsf{P}(\omega_2) = 1/2$. Consider the random sequence defined by

$$\begin{aligned} X_0(\omega_1) &= 1, & X_0(\omega_2) &= 1, \\ X_1(\omega_1) &= 1, & X_1(\omega_2) &= 3/2, \\ X_2(\omega_1) &= 4/5, & X_2(\omega_2) &= 5/3. \end{aligned}$$

Let $a = b = 4/5$. One can check that (4.4) is satisfied for any deterministic times $u \le v$. On the other hand, this property is violated for the stopping times $u = 0$, $v(\omega_1) = 1$, $v(\omega_2) = 2$. □

The following remark clarifies the nature of delta-martingales.

`Remark.` Let $(X_t)_{t\in[0,T]}$ be an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted càdlàg process with $\mathsf{E}|X_T| < \infty$. Set $\widetilde{X}_t = \mathsf{E}(X_T \,|\, \mathcal{F}_t)$, $t \in [0,T]$. If X is an $(\mathcal{F}_t, a, b)$-delta-martingale, then, clearly, $b\widetilde{X} \le X \le a^{-1}\widetilde{X}$. "Conversely", if $c\widetilde{X} \le X \le c^{-1}\widetilde{X}$, where $c = a^{1/2} \vee b^{1/2}$, then, for any $(\mathcal{F}_t)$-stopping times $u \le v$, we have

$$c\widetilde{X}_u \le X_u \le c^{-1}\widetilde{X}_u,$$
$$c\widetilde{X}_u = c\mathsf{E}(\widetilde{X}_v \,|\, \mathcal{F}_u) \le \mathsf{E}(X_v \,|\, \mathcal{F}_u) \le c^{-1}\mathsf{E}(\widetilde{X}_v \,|\, \mathcal{F}_u) = c^{-1}\widetilde{X}_u,$$

which implies that

$$aX_u \le c^2 X_u \le \mathsf{E}(X_v \,|\, \mathcal{F}_u) \le c^{-2}X_u \le b^{-1}X_u,$$

i.e. X is an $(\mathcal{F}_t, a, b)$-delta-martingale.

The following corollary of Lemma 4.7 yields a convenient description of delta-martingales of order (a, a).

Corollary 4.11. *Let $a \in [0,1]$. An $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted càdlàg process X is an $(\mathcal{F}_t, a, a)$-delta-martingale if and only if there exists an $(\mathcal{F}_t)$-martingale M such that $aX \le M \le X$.*

`Notation.` Set

$$\mathcal{M} = \left\{\mathsf{Q} \sim \mathsf{P} : \overline{S}^i \text{ is an } (\mathcal{F}_t, \mathsf{Q}, (1-\alpha^i-\beta^i), (1-\alpha^i))\text{-delta-martingale}, i = 1, ..., d\right\}.$$

`Important remark.` Consider a model with no transaction costs and with short selling prohibited, i.e. a model with $\alpha^i = 0$, $\beta^i = 1$, $i = 1, \dots, d$. Then

$$\mathcal{M} = \left\{\mathsf{Q} \sim \mathsf{P} : \overline{S}^i \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-supermartingale}, i = 1, \dots, d\right\}.$$

Key Lemma 4.12. *For the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}(Z_0) = \mathcal{M}.$$

`Proof.` This statement is a direct consequence of Key Lemma 4.3. □

4.3 Model with European Options

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $T \in \mathbb{R}_{++}$. Let S_T be an $\mathbb{R}_+^d$-valued random vector on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, S_T^i is the ask price of the i-th asset at time T (assets $1, \dots, d$ are the same as in Section 2.1). For simplicity, we consider only proportional transaction costs on the underlying assets, i.e. the bid price of the i-th asset at time T is $(1-\alpha^i)S_T^i$, where $\alpha^i \in [0,1)$. For $i = 1, \dots, d$, let $\mathbb{K}^i \subseteq \mathbb{R}_+$ be the set of strike prices K of traded European call options on the i-th asset with maturity T. Let $\varphi^{ai}(K)$ and $\varphi^{bi}(K)$, $i = 1, \dots, d$, $K \in \mathbb{K}^i$ be the ask and bid prices at time 0 of such an option. (The short position in the option can be taken without paying additional premium for short selling.) Let $r \in \mathbb{R}_+$ be the risk-free interest rate. Define the set of attainable incomes by

$$A = \biggl\{\sum_{n=1}^{N}\sum_{i=1}^{d}\biggl[g_n^i\biggl(\frac{((1-\alpha^i)S_T^i - K_n^i)^+}{1+r} - \varphi^{ai}(K_n^i)\biggr) + h_n^i\biggl(-\frac{(S_T^i - K_n^i)^+}{1+r} + \varphi^{bi}(K_n^i)\biggr)\biggr] : \\ N \in \mathbb{N},\ K_n^i \in \mathbb{K}^i,\ g_n^i, h_n^i \in \mathbb{R}_+\biggr\}.$$

The possibility to trade the underlying assets at time 0 is easily incorporated into the model by assuming that $0 \in \mathbb{K}^i$, $i = 1, \dots, d$.

`Notation`. Set

$$\mathcal{D}^i = \biggl\{\mu : \mu \text{ is a probability measure on } \mathcal{B}(\mathbb{R}_+) \text{ such that} \\ \int_{\mathbb{R}_+} ((1-\alpha^i)x - K)^+ \mu(dx) \le (1+r)\varphi^{ai}(K) \text{ and} \\ \int_{\mathbb{R}_+} (x-K)^+ \mu(dx) \ge (1+r)\varphi^{bi}(K),\ K \in \mathbb{K}^i\biggr\}, \quad i = 1, \dots, d.$$

`Important remark`. By Lemma 3.23, an equivalent description of $\mathcal{D}^i$ is as follows:

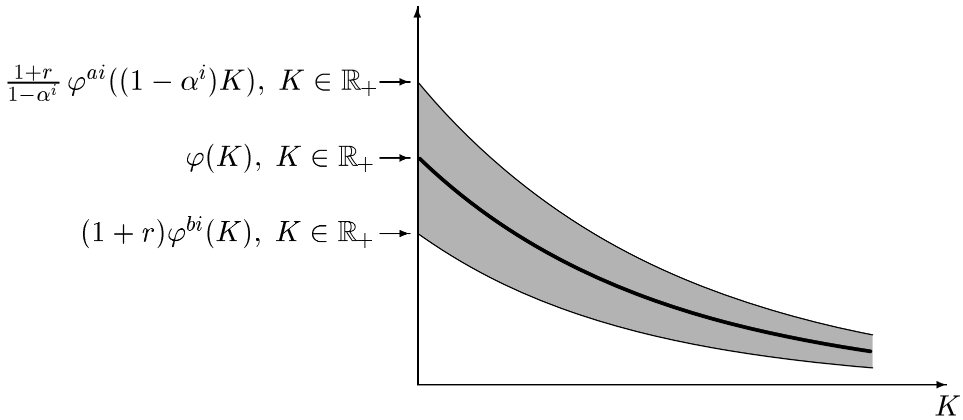
$$\mathcal{D}^i = \biggl\{\varphi'' : \varphi \text{ is convex on } \mathbb{R}_+,\ \varphi'_+(0) \ge -1,\ \lim_{K\to\infty} \varphi(K) = 0, \\ \varphi((1-\alpha^i)^{-1}K) \le \frac{1+r}{1-\alpha^i}\varphi^{ai}(K), \text{ and } \varphi(K) \ge (1+r)\varphi^{bi}(K),\ K \in \mathbb{K}^i\biggr\}.$$

`Notation`. Set $\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \mathrm{Law}_{\mathsf{Q}}\, S_T^i \in \mathcal{D}^i,\ i = 1, \dots, d\}$.

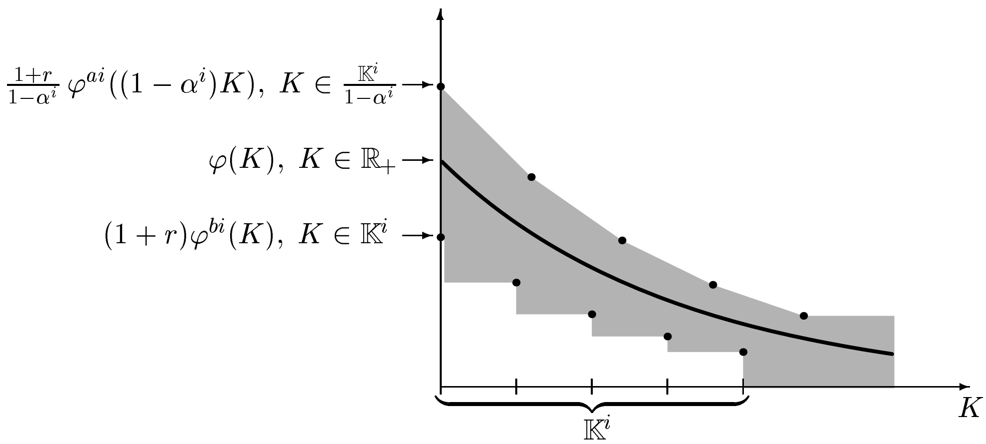
Key Lemma 4.13. *Suppose that* $0 \in \mathbb{K}^i$, $i = 1, \dots, d$. *Then, for the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}\biggl(\sum_{i=1}^{d}((1-\alpha^i)S_T^i - (1+r)\varphi^{ai}(0))\biggr) = \mathcal{M}.$$

This statement is clear (take Lemma 2.10 into account).



`Figure 7.a.` The structure of $\mathcal{D}^i$ in the case, where $\mathbb{K}^i = \mathbb{R}_+$. The set $\mathcal{D}^i$ consists of the second derivatives φ'', where φ is convex on $\mathbb{R}_+$, $\varphi'_+(0) \geq -1$, $\lim_{K\to\infty} \varphi(K) = 0$, and φ lies in the shaded region.



`Figure 7.b.` The structure of $\mathcal{D}^i$ in the case, where $\mathbb{K}^i$ is finite. The set $\mathcal{D}^i$ consists of the second derivatives φ'', where φ is convex on $\mathbb{R}_+$, $\varphi'_+(0) \geq -1$, $\lim_{K\to\infty} \varphi(K) = 0$, and φ lies in the shaded region.

4.4 Mixed Model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial and $(\mathcal{F}_t)$ is right-continuous. Let $(S_t)_{t\in[0,T]}$ be an $\mathbb{R}^d_+$-valued $(\mathcal{F}_t)$-adapted càdlàg process. From the financial point of view, S^i_t is the ask price of the i-th asset at time t (assets $1, \dots, d$ are the same as in Section 2.1). For simplicity, we consider only proportional transaction costs and costs of short selling on the underlying assets, i.e. the bid price of the i-th asset at time t is $(1-\alpha^i)S^i_t$, while the amount obtained by short selling this asset at time t is $(1-\alpha^i-\beta^i)S^i_t$, where $\alpha^i \in [0,1)$, $\beta^i \in [0,1]$, $\alpha^i + \beta^i \leq 1$. For $i = 1, \dots, d$, let $\mathbb{E}^i \subseteq [0,T] \times \mathbb{R}_+$ be the set of pairs (t, K), for which there exists a traded European call option on the i-th asset with maturity t and strike price K. Let $\varphi^{ai}(t,K)$ and $\varphi^{bi}(t,K)$, $i = 1, \dots, d$, $(t,K) \in \mathbb{E}^i$ be the ask and bid prices at time 0 of such an option. We assume that the set of possible maturities

$$\{t \in [0,T] : \text{there exist } i, K \text{ such that } (t,K) \in \mathbb{E}^i\}$$

is countable. Let $r \in \mathbb{R}_+$ be the risk-free interest rate. Define $\overline{S}$ by (3.2). Define the set of attainable incomes by

$$\begin{aligned} A = \Bigg\{ & \sum_{m=1}^{M}\sum_{i=1}^{d} \big[G_m^i\big((1-\alpha^i)\overline{S}^i_{v_m} - \overline{S}^i_{u_m}\big) + H_m^i\big(-\overline{S}^i_{v_m} + (1-\alpha^i-\beta^i)\overline{S}^i_{u_m}\big)\big] \\ & + \sum_{n=1}^{N}\sum_{i=1}^{d} \big[g_n^i\big(e^{-rt_n^i}\big((1-\alpha^i)S^i_{t_n^i} - K_n^i\big)^+ - \varphi^{ai}(t_n^i, K_n^i)\big) \\ & + h_n^i\big(-e^{-rt_n^i}\big(S^i_{t_n^i} - K_n^i\big)^+ + \varphi^{bi}(t_n^i, K_n^i)\big)\big] : M, N \in \mathbb{N}, \\ & u_m \le v_m \text{ are } (\mathcal{F}_t)\text{-stopping times, } G_m^i, H_m^i \text{ are } \mathbb{R}_+\text{-valued and} \\ & \mathcal{F}_{u_m}\text{-measurable, } (t_n^i, K_n^i) \in \mathbb{E}^i,\ g_n^i, h_n^i \in \mathbb{R}_+ \Bigg\}. \end{aligned}$$

`Notation.` (i) Set

$$\mathbb{K}^i(t) = \{K \in \mathbb{R}_+ : (t,K) \in \mathbb{E}^i\}, \quad i = 1,\dots,d,\ t \in [0,T],$$

$$\begin{aligned} \mathcal{D}^i(t) = \Bigg\{ & \mu : \mu \text{ is a probability measure on } \mathcal{B}(\mathbb{R}_+) \text{ such that} \\ & \int_{\mathbb{R}_+} ((1-\alpha^i)x - K)^+ \mu(dx) \le e^{rt}\varphi^{ai}(t,K) \text{ and} \\ & \int_{\mathbb{R}_+} (x-K)^+ \mu(dx) \ge e^{rt}\varphi^{bi}(t,K),\ K \in \mathbb{K}^i(t) \Bigg\}, \quad i = 1,\dots,d,\ t \in [0,T], \end{aligned}$$

$$\begin{aligned} \mathcal{M} = \big\{ & \mathsf{Q} \sim \mathsf{P} : \overline{S}^i \text{ is an } (\mathcal{F}_t, \mathsf{Q}, (1-\alpha^i-\beta^i)^+, 1-\alpha^i)\text{-delta martingale,} \\ & i = 1,\dots,d \text{ and } \operatorname{Law}_{\mathsf{Q}} S_t^i \in \mathcal{D}^i(t),\ i = 1,\dots,d,\ t \in [0,T]\big\}. \end{aligned}$$

(ii) Let $t_1, t_2, \dots$ be a numbering of

$$\{t \in [0,T] : \text{there exist } i, K \text{ such that } (t,K) \in \mathbb{E}^i\} \cup \{qT : q \in \mathbb{Q} \cap [0,1]\}$$

and set

$$Z_0 = \sum_{n=1}^{\infty}\sum_{i=1}^{d} 2^{-n}\big((1-\alpha^i)\overline{S}^i_{t_n} - \overline{S}^i_0\big).$$

Key Lemma 4.14. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}(Z_0) = \mathcal{M}.$$

`Proof.` *Step 1.* The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 2.10.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(Z_0)$. Fix $i \in \{1,\dots,d\}$. The proof of Key Lemma 4.7 (Step 2) shows that $\overline{S}^i$ is an $(\mathcal{F}_t, \mathsf{Q})$-delta-martingale of order $((1-\alpha^i-\beta^i)^+, 1-\alpha^i)$. Fix $(t,K) \in \mathbb{E}^i$. There exist $\lambda, \rho \in \mathbb{R}_+$ such that

$$-(S_t^i - K)^+ + e^{rt}\varphi^{bi}(t,K) \ge -\lambda Z_0 - \rho.$$

Hence,

$$\mathsf{E}_{\mathsf{Q}}(S_t^i - K)^+ \geq e^{rt}\varphi^{bi}(t, K).$$

In a similar way, we obtain

$$\mathsf{E}_{\mathsf{Q}}((1-\alpha^i)S_t^i - K)^+ \leq e^{rt}\varphi^{ai}(t, K).$$

As a result, $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Fix $X = X_1 + X_2 \in A$, where

$$\begin{aligned} X_1 &= \sum_{m=1}^{M}\sum_{i=1}^{d}\big[G_m^i\big((1-\alpha^i)\overline{S}_{v_m}^i - \overline{S}_{u_m}^i\big) + H_m^i\big(-\overline{S}_{v_m}^i + (1-\alpha^i-\beta^i)\overline{S}_{u_m}^i\big)\big],\\ X_2 &= \sum_{n=1}^{N}\sum_{i=1}^{d}\big[g_n^i\big(e^{-rt_n^i}\big((1-\alpha^i)S_{t_n^i}^i - K_n^i\big)^+ - \varphi^{ai}(t_n^i, K_n^i)\big)\\ &\quad + h_n^i\big(-e^{-rt_n^i}\big(S_{t_n^i}^i - K_n^i\big)^+ + \varphi^{bi}(t_n^i, K_n^i)\big)\big]. \end{aligned}$$

Clearly, X_2 is Q-integrable and $\mathsf{E}_{\mathsf{Q}}X_2 \leq 0$. The proof of Key Lemma 4.7 (Step 3) shows that $\mathsf{E}_{\mathsf{Q}}X^- \geq \mathsf{E}_{\mathsf{Q}}X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

5 Possibility Approach

In this chapter, we introduce the possibility approach to arbitrage pricing.

Section 5.1 contains several examples known in financial mathematics of pricing with no use of probability measure. These examples serve as prototypes of the possibility approach.

Sections 5.2–5.5 are the “possibility duals” of Sections 2.1–2.4.

Section 5.6 describes how to obtain the “possibility duals” of the models of Chapters 3, 4.

As an example, we consider in Section 5.9 the “possibility dual” of the model of Section 3.4.

In Sections 5.7, 5.8, and 5.10, we study whether the possibility version of the NGA condition is satisfied for some models of Chapters 3, 4.

	Probability approach	**Possibility approach**
Pricing by the ordinary arbitrage in the one-period model	2.1	5.2
Pricing by the generalized arbitrage in the general arbitrage pricing model	2.2, 2.3, 2.4	5.3, 5.4, 5.5

Table 4. The structure table for various approaches to pricing by arbitrage. The numbers indicate the sections, in which the corresponding approach is described.

5.1 Examples

We begin with a probability example of calculating the fair price in a one-period binomial model.

Example 5.1. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $a, b \in \mathbb{R}_+$, $\overline{S}_0 \in (a, b)$, $\overline{S}_1$ be an $\{a, b\}$-valued random variable on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, $\overline{S}_n$ is the discounted price of some asset at time n. Consider a contingent claim $F = f(\overline{S}_1)$. Define the fair price of F by

$$C(F) = \{x : \text{there exists } h \in \mathbb{R} \text{ such that } x + h(\overline{S}_1 - \overline{S}_0) = F \ \mathsf{P}\text{-a.s.}\}.$$

One can check that if $\mathsf{P}(\overline{S}_1 = a) > 0$ and $\mathsf{P}(\overline{S}_1 = b) > 0$, then

$$C(F) = \frac{b - \overline{S}_0}{b - a} f(a) + \frac{\overline{S}_0 - a}{b - a} f(b). \tag{5.1}$$

The next example shows that the same result can be obtained with no use of the original probability measure.

Example 5.2. Let Ω, $\mathcal{F}$, $\overline{S}_0$, $\overline{S}_1$, F be the same as in the previous example. Define the fair price of F by

$$C(F) = \{x : \text{there exists } h \in \mathbb{R} \text{ such that } x + h(\overline{S}_1 - \overline{S}_0) = F \text{ pointwise}\}.$$

One can check that if $\{\overline{S}_1(\omega) : \omega \in \Omega\} = \{a, b\}$, then (5.1) is true. □

`Remark.` In the multiperiod binomial model (known as the Cox-Ross-Rubinstein model), the fair price of a contingent claim can also be calculated with no use of the original probability measure.

We now pass on to a more general one-period model.

Example 5.3. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\overline{S}_0 \in \mathbb{R}_{++}$, $\overline{S}_1$ be an $\mathbb{R}_{++}$-valued random variable on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, $\overline{S}_n$ is the discounted price of some asset at time n. Consider the contingent claim $F = (\overline{S}_1 - K)^+$, where $K \in \mathbb{R}_+$ (we assume for simplicity that the risk-free interest rate is zero, so that F is the European call option). Define the lower and upper prices of F by

$$C_*(F) = \sup\{x : \text{there exists } h \in \mathbb{R} \text{ such that } x + h(\overline{S}_1 - \overline{S}_0) \le F \ \mathsf{P}\text{-a.s.}\},$$
$$C^*(F) = \inf\{x : \text{there exists } h \in \mathbb{R} \text{ such that } x + h(\overline{S}_1 - \overline{S}_0) \ge F \ \mathsf{P}\text{-a.s.}\}.$$

One can check that if $\operatorname{supp} \operatorname{Law}_{\mathsf{P}} \overline{S}_1 = \mathbb{R}_+$, then

$$C_*(F) = (\overline{S}_0 - K)^+, \qquad C^*(F) = \overline{S}_0. \tag{5.2}$$

The next example shows that the same result can be obtained with no use of the original probability measure. This example is borrowed from [S99; Ch. V, § 1c] (to be more precise, this book contains a similar example).

Example 5.4. Let Ω, $\mathcal{F}$, $\overline{S}_0$, $\overline{S}_1$, F be the same as in the previous example. Define the lower and upper prices of F by

$$C_*(F) = \sup\{x : \text{there exists } h \in \mathbb{R} \text{ such that } x + h(\overline{S}_1 - \overline{S}_0) \le F \text{ pointwise}\},$$
$$C^*(F) = \inf\{x : \text{there exists } h \in \mathbb{R} \text{ such that } x + h(\overline{S}_1 - \overline{S}_0) \ge F \text{ pointwise}\}.$$

One can check that if $\{\overline{S}_1(\omega) : \omega \in \Omega\} = \mathbb{R}_{++}$, then (5.2) is true. □

5.2 Ordinary Arbitrage in a One-Period Model

Definition 5.5. A *possibility space* is a pair $(\Omega,\mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a σ-field on Ω. We call Ω the *set of possible elementary events.*

Let $(\Omega,\mathcal{F})$ be a possibility space. Let $S_0\in\mathbb{R}^d$ and S_1 be an $\mathbb{R}^d$-valued $\mathcal{F}$-measurable function. From the financial point of view, S_n^i is the price of the i-th asset at time n (assets $1,\dots,d$ are the same as in Section 2.1). Define $\overline{S}$ by (2.1). Define the set of attainable incomes by

$$A=\biggl\{\sum_{i=1}^d h^i\bigl(\overline{S}_1^i-\overline{S}_0^i\bigr):h^i\in\mathbb{R}\biggr\}.$$

Definition 5.6. A *one-period model* is a collection $(\Omega,\mathcal{F},\overline{S}_0,\overline{S}_1)$.

Definition 5.7. The model $(\Omega,\mathcal{F},\overline{S}_0,\overline{S}_1)$ satisfies the *no arbitrage* (NA) condition if $A\cap L_+^0=\{0\}$ (L_+^0 denotes the set of $\mathbb{R}_+$-valued $\mathcal{F}$-measurable functions).

Definition 5.8. A *martingale measure* is a probability measure Q on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}}|\overline{S}_1|<\infty$ and $\mathsf{E}_{\mathsf{Q}}\overline{S}_1=\overline{S}_0$. The set of martingale measures will be denoted by $\mathcal{M}$.

`Notation`. Set $C=\overline{\operatorname{conv}}\{\overline{S}_1(\omega):\omega\in\Omega\}$ and let C° denote the relative interior of C.

Theorem 5.9 (FTAP). *For the model $(\Omega,\mathcal{F},\overline{S}_0,\overline{S}_1)$, the following conditions are equivalent:*

(a) *NA;*
(b) $\overline{S}_0\in C^\circ$;
(c) *for any $D\in\mathcal{F}\setminus\{\emptyset\}$, there exists $\mathsf{Q}\in\mathcal{M}$ such that $\mathsf{Q}(D)>0$.*

`Remark`. Each of conditions (a)–(c) of Theorem 2.4, which is the "probability dual" of Theorem 5.9, is equivalent to the following one:

(c') for any $D\in\mathcal{F}$ such that $\mathsf{P}(D)>0$, there exists a measure $\mathsf{Q}\ll\mathsf{P}$ such that $\mathsf{E}_{\mathsf{Q}}|\overline{S}_1|<\infty$, $\mathsf{E}_{\mathsf{Q}}\overline{S}_1=\overline{S}_0$, and $\mathsf{Q}(D)>0$.

Indeed, the implication (c)$\Rightarrow$(c') is trivial, while the implication (c')$\Rightarrow$(a) is straightforward.

`Proof of Theorem 5.9`. *Step 1.* Let us prove the implication (a)$\Rightarrow$(b). If $\overline{S}_0\notin C^\circ$, then, by the separation theorem, there exists $h\in\mathbb{R}^d$ such that $\langle h,(\overline{S}_1-\overline{S}_0)\rangle\ge 0$ pointwise and $\langle h,(\overline{S}_1(\omega)-\overline{S}_0(\omega))\rangle>0$ for some $\omega\in\Omega$. This contradicts the NA condition.

Step 2. Let us prove the implication (b)$\Rightarrow$(c). Fix $D\in\mathcal{F}\setminus\{\emptyset\}$. Take $\omega_0\in D$. The set

$$E=\biggl\{\sum_{k=0}^m\alpha_k\overline{S}_1(\omega_k):m\in\mathbb{N},\ \omega_1,\dots,\omega_m\in\Omega,\ \alpha_0,\dots,\alpha_m\in\mathbb{R}_{++},\ \sum_{k=0}^m\alpha_k=1\biggr\}$$

is convex, and the closure of E contains $\{\overline{S}_1(\omega):\omega\in\Omega\}$. Consequently, $E\supseteq C^\circ$. Thus, there exist $\omega_1,\dots,\omega_m\in\Omega$ and $\alpha_0,\dots,\alpha_m\in\mathbb{R}_{++}$ such that $\sum_{k=0}^m\alpha_k=1$ and $\sum_{k=0}^m\alpha_k\overline{S}_1(\omega_k)=\overline{S}_0$. Then the measure $\mathsf{Q}=\sum_{k=0}^m\alpha_k\delta_{\omega_k}$ belongs to $\mathcal{M}$ and $\mathsf{Q}(D)>0$.

Step 3. Let us prove the implication (c)$\Rightarrow$(a). Suppose that the NA condition is not satisfied, i.e. there exists $X \in A \cap (L^0_+ \setminus \{0\})$. Consider $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(X > 0) > 0$. Then $\mathsf{E_Q} X > 0$. On the other hand, as $\mathsf{Q} \in \mathcal{M}$, we should have $\mathsf{E_Q} X = 0$. The obtained contradiction shows that the NA condition is satisfied. □

Now, let F be a real-valued $\mathcal{F}$-measurable function. From the financial point of view, F is the payoff of some contingent claim discounted to time 0.

Definition 5.10. (i) A real number x is a *fair price* of F if the model with $d+1$ assets $(\Omega, \mathcal{F}, x, \overline{S}^1_0, \dots, \overline{S}^d_0, F, \overline{S}^1_1, \dots, \overline{S}^d_1)$ satisfies the NA condition. The set of fair prices of F will be denoted by $I(F)$.

(ii) The *lower* and *upper* prices of F are defined by

$$V_*(F) = \inf\{x : x \in I(F)\},$$
$$V^*(F) = \sup\{x : x \in I(F)\}.$$

`Notation`. Set $D = \overline{\text{conv}}\{(F(\omega), \overline{S}_1(\omega)) : \omega \in \Omega\}$ and let D° denote the relative interior of D.

Theorem 5.11. *Suppose that the model $(\Omega, \mathcal{F}, \overline{S}_0, \overline{S}_1)$ satisfies the NA condition. Then*

$$I(F) = \{x : (x, \overline{S}_0) \in D^\circ\} \approx \{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{M}\}, \tag{5.3}$$
$$V_*(F) = \inf_{\mathsf{Q} \in \mathcal{M}} \mathsf{E_Q} F, \tag{5.4}$$
$$V^*(F) = \sup_{\mathsf{Q} \in \mathcal{M}} \mathsf{E_Q} F. \tag{5.5}$$

The expectation $\mathsf{E_Q} F$ here is taken in the sense of finite expectations, i.e. we consider only those Q, for which $\mathsf{E_Q}|F| < \infty$.

`Proof`. One should prove only (5.3). Theorem 5.9 implies that

$$I(F) = \{x : (x, \overline{S}_0) \in D^\circ\} \subseteq \{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{M}\}. \tag{5.6}$$

Let $x \in \{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{M}\}$. Take $\mathsf{Q}_0 \in \mathcal{M}$ such that $x = \mathsf{E}_{\mathsf{Q}_0} F$. One can find $\mathsf{Q}_1 \in \mathcal{M}$ such that $\mathsf{E}_{\mathsf{Q}_1}|F| < \infty$ and $\overline{\text{conv}}\,\text{supp}\,\text{Law}_{\mathsf{Q}_1}(F, \overline{S}_1) = D$ (Q_1 can be found in the form $\sum_{n=1}^\infty \alpha_n \delta_{\omega_n}$). For any $\varepsilon \in (0,1)$, the measure $\mathsf{Q}(\varepsilon) = (1-\varepsilon)\mathsf{Q}_0 + \varepsilon \mathsf{Q}_1$ belongs to $\mathcal{M}$ and $\overline{\text{conv}}\,\text{supp}\,\text{Law}_{\mathsf{Q}(\varepsilon)}(F, \overline{S}_1) = D$. Therefore, $\mathsf{E}_{\mathsf{Q}(\varepsilon)}(F, \overline{S}_1) \in D^\circ$, which means that

$$\mathsf{E}_{\mathsf{Q}(\varepsilon)} F \in \{x : (x, \overline{S}_0) \in D^\circ\}.$$

Furthermore, $\mathsf{E}_{\mathsf{Q}(\varepsilon)} F \underset{\varepsilon \downarrow 0}{\longrightarrow} x$. This, together with (5.6), proves the approximate equality in (5.3). □

`Remarks`. `(i)` Let F be such that $V_*(F) < V^*(F)$. It follows from the equality $I(F) = \{x : (x, \overline{S}_0) \in D^\circ\}$ that $I(F) = (V_*(F), V^*(F))$. As for the interval $\{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{M}\}$, it has the endpoints $V_*(F)$ and $V^*(F)$, but may contain them. For instance, this interval contains $V^*(F)$ if and only if

$$(V^*(F), \overline{S}_0) \in \text{conv}\{(F(\omega), \overline{S}_1(\omega)) : \omega \in \Omega\}.$$

(ii) Another way to define the lower and upper prices is as follows:

$$C_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \le F \text{ pointwise}\},$$
$$C^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \ge F \text{ pointwise}\}.$$

Using the equality $I(F) = \{x : (x, \overline{S}_0) \in D^\circ\}$ and elementary geometric considerations, one can check that if the model $(\Omega, \mathcal{F}, \overline{S}_0, \overline{S}_1)$ satisfies the NA condition, then $C_*(F) = V_*(F)$ and $C^*(F) = V^*(F)$.

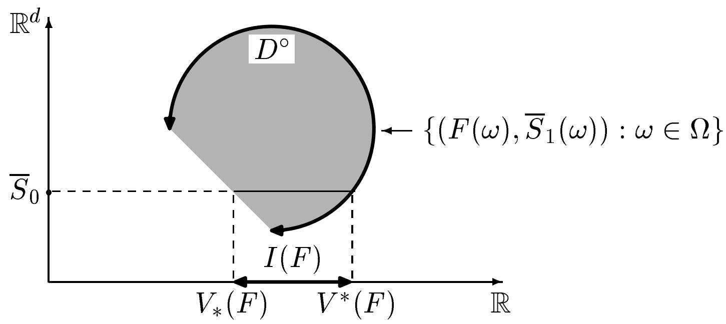


Figure 8. The joint arrangement of $I(F)$, $V_*(F)$, $V^*(F)$, $\{\mathsf{E}_\mathsf{Q}F : \mathsf{Q} \in \mathcal{M}\}$, and D°. In the example shown here, $I(F) = (V_*(F), V^*(F))$, while $\{\mathsf{E}_\mathsf{Q}F : \mathsf{Q} \in \mathcal{M}\} = (V_*(F), V^*(F)]$.

5.3 Generalized Arbitrage

Let $(\Omega, \mathcal{F})$ be a possibility space.

Definition 5.12. An *arbitrage pricing model* is a triple $(\Omega, \mathcal{F}, A)$, where A is a convex cone in L^0 (L^0 is the space of real-valued $\mathcal{F}$-measurable functions). The set A will be called the *set of attainable incomes*.

Notation. (i) Set

$$B = \Big\{ Z \in L^0 : \text{ there exist } (X_n)_{n \in \mathbb{N}} \in A \text{ and } a \in \mathbb{R} \text{ such that } X_n \ge a \text{ pointwise and } Z = \lim_{n\to\infty} X_n \text{ pointwise} \Big\} \tag{5.7}$$

(ii) For $Z \in B$, denote $\gamma(Z) = 1 - \inf_{\omega\in\Omega} Z(\omega)$ and set

$$\begin{aligned} A_1 &= \{X - Y : X \in A,\ Y \in L^0_+\},\\ A_2(Z) &= \Big\{ \frac{X}{Z + \gamma(Z)} : X \in A_1 \Big\},\\ A_3(Z) &= A_2(Z) \cap L^\infty,\\ A_4(Z) &= \text{closure of } A_3(Z) \text{ in } \sigma(L^\infty, M_F). \end{aligned} \tag{5.8}$$

Here L^0_+ is the set of $\mathbb{R}_+$-valued elements of L^0; L^∞ is the space of bounded elements of L^0; $\sigma(L^\infty, M_F)$ denotes the weak topology on L^∞ induced by the space M_F of finite σ-additive measures on $\mathcal{F}$ (i.e. signed measures with finite variation).

Definition 5.13. The model $(\Omega, \mathcal{F}, A)$ satisfies the *no generalized arbitrage* (NGA) condition if for any $Z \in B$, we have $A_4(Z) \cap L^0_+ = \{0\}$.

Remark. One can define the *no arbitrage* (NA) condition as: $A \cap L^0_+ = \{0\}$.

Definition 5.14. A *risk-neutral measure* is a probability measure Q on $\mathcal{F}$ such that $\mathsf{E}_\mathsf{Q} X^- \geq \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$. The expectations $\mathsf{E}_\mathsf{Q} X^-$ and $\mathsf{E}_\mathsf{Q} X^+$ here may take on the value $+\infty$. The set of risk-neutral measures will be denoted by $\mathcal{R}$.

Notation. For $Z \in B$, we will denote by $\mathcal{R}(Z)$ the set of probability measures Q on $\mathcal{F}$ with the property: for any $X \in A$ such that $X \geq -\alpha Z - \beta$ pointwise with some $\alpha, \beta \in \mathbb{R}_+$, we have $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \leq 0$.

Lemma 5.15. *For any $Z \in B$, we have $\mathcal{R} \subseteq \mathcal{R}(Z)$.*

This statement is proved in the same way as Lemma 2.10.

The following basic assumption is satisfied in all the particular models considered below.

Assumption 5.16. There exists $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$ (in particular, both sets might be empty).

Theorem 5.17 (FTAP). *Suppose that Assumption 5.16 is satisfied. Then the model $(\Omega, \mathcal{F}, A)$ satisfies the NGA condition if and only if for any $D \in \mathcal{F} \setminus \{\emptyset\}$, there exists a risk-neutral measure Q such that $\mathsf{Q}(D) > 0$.*

Remark. Each of the equivalent conditions of Theorem 2.12, which is the "probability dual" of Theorem 5.17, is equivalent to the following one: for any $D \in \mathcal{F}$ such that $\mathsf{P}(D) > 0$, there exists $\mathsf{Q} \ll \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X^- \geq \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$ and $\mathsf{Q}(D) > 0$ (this is proved similarly to Theorem 2.12).

The proof of Theorem 5.17 is based on the following statement (the possibility analogue of the Kreps–Yan theorem):

Lemma 5.18. *Let C be a $\sigma(L^\infty, M_F)$-closed convex cone in L^∞ such that $C \supseteq L^\infty_-$ (L^∞_- is the set of negative elements of L^∞). Let $W \in L^\infty \setminus C$. Then there exists a probability measure Q on $\mathcal{F}$ such that $\mathsf{E}_\mathsf{Q} X \leq 0$ for any $X \in C$ and $\mathsf{E}_\mathsf{Q} W > 0$.*

Proof. By the Hahn-Banach separation theorem (see [S71; Ch. II, Th. 9.2]), there exists a measure $\mathsf{Q}_0 \in M_F$ such that $\mathsf{E}_{\mathsf{Q}_0} W \notin \{\mathsf{E}_{\mathsf{Q}_0} X : X \in C\}$. Without loss of generality, $\mathsf{E}_{\mathsf{Q}_0} W > 0$. As C is a cone, $\mathsf{E}_{\mathsf{Q}_0} X \leq 0$ for any $X \in C$. Since $C \supseteq L^\infty_-$, Q_0 is positive. Then the measure $\mathsf{Q} = c\mathsf{Q}_0$, where c is the normalizing constant, satisfies the desired properties. □

Proof of Theorem 5.17. *Step 1.* Let us prove the "only if" implication. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Set $W = I_D$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Lemma 5.17 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A_4(Z_0)$ and to the point W yields a probability measure Q_0 on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \leq 0$ for any $X \in A_4(Z_0)$ and $\mathsf{E}_{\mathsf{Q}_0} W > 0$. By the Fatou lemma, for any $X \in A$ such that $\frac{X}{Z_0 + \gamma(Z_0)}$ is bounded below, we have

$\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_0+\gamma(Z_0)} \le 0$. Consider the measure $\mathsf{Q} = \frac{c}{Z_0+\gamma(Z_0)} \mathsf{Q}_0$, where c is the normalizing constant. Then $\mathsf{Q} \in \mathcal{R}(Z_0) = \mathcal{R}$ and

$$\mathsf{Q}(D) = \mathsf{E}_{\mathsf{Q}_0} \frac{cW}{Z_0 + \gamma(Z_0)} > 0.$$

Step 2. Let us prove the "if" implication. Suppose that the NGA condition is not satisfied. Then there exist $Z \in B$ and $W \in A_4(Z) \cap (L^0_+ \setminus \{0\})$. Take $\mathsf{Q} \in \mathcal{R}$ such that $\mathsf{Q}(W > 0) > 0$. It follows from the Fatou lemma that Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where c is the normalizing constant. For any $X \in A$ such that $\frac{X}{Z+\gamma(Z)}$ is bounded below by a constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E}_{\mathsf{Q}} X^- \le \mathsf{E}_{\mathsf{Q}}(\alpha Z + \alpha\gamma(Z)) < \infty,$$

and consequently,

$$\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{X}{Z + \gamma(Z)} = c\mathsf{E}_{\mathsf{Q}} X \le 0.$$

Hence, $\mathsf{E}_{\widetilde{\mathsf{Q}}} X \le 0$ for any $X \in A_4(Z)$. On the other hand,

$$\mathsf{E}_{\widetilde{\mathsf{Q}}} W = c\mathsf{E}_{\mathsf{Q}}(Z + \gamma(Z))W > 0.$$

The obtained contradiction shows that the NGA condition is satisfied. □

It is seen from the above proof that the necessity part in Theorem 5.17 is true without Assumption 5.16. The following example shows that this assumption is essential for the sufficiency part.

Example 5.19. Let

$$\Omega = \{\omega : \omega \text{ is a real-valued } \mathcal{B}(\mathbb{R}_{++})\text{-measurable function such that } \omega(t) = 1 \text{ for all } t \in \mathbb{R}_{++}, \text{ except for a finite set}\}.$$

Set $X_t(\omega) = \omega(t)$. Consider an auxiliary process

$$Y_t(\omega) = \sum_{s \le t} I(X_s(\omega) \ne 1), \quad \omega \in \Omega,\ t \in \mathbb{R}_{++}$$

and set $\mathcal{F} = \sigma(X_t, Y_t; t \in \mathbb{R}_{++})$,

$$A = \left\{ \sum_{n=1}^{N} h_n X_{t_n} : N \in \mathbb{N},\ t_n \in \mathbb{R}_{++},\ h_n \in \mathbb{R} \right\}.$$

Clearly, the only element of A that is bounded below is 0. This implies that the model $(\Omega, \mathcal{F}, A)$ satisfies the NGA condition.

Suppose now that there exists a risk-neutral measure Q. Consider the set

$$C = \{(\omega, t) \in \Omega \times \mathbb{R}_{++} : X_t(\omega) \ne 1\}$$

and, for $n \in \mathbb{N}$, define the set

$$C_n = \left\{ (\omega, t) \in \Omega \times \mathbb{R}_{++} : X_s(\omega) \ne 1 \text{ for some } s \in \left(\frac{[tn]}{n}, \frac{[tn]+1}{n} \right] \right\},$$

where $[tn]$ denotes the integer part of tn. In view of the equality

$$C_n = \bigcup_{m=0}^{\infty} \left\{ (\omega, t) \in \Omega \times \mathbb{R}_{++} : t \in \left(\frac{m}{n}, \frac{m+1}{n} \right] \text{ and } Y_{\frac{m+1}{n}}(\omega) - Y_{\frac{m}{n}}(\omega) > 0 \right\},$$

we have $C_n \in \mathcal{F} \times \mathcal{B}(\mathbb{R}_{++})$, and consequently, $C = \bigcap_{n=1}^{\infty} C_n \in \mathcal{F} \times \mathcal{B}(\mathbb{R}_{++})$. Consider the functions

$$f_n(\omega) = \int_{\mathbb{R}_{++}} I((\omega, t) \in C_n) dt, \quad \omega \in \Omega, \ n \in \mathbb{N},$$

$$g_n(t) = \int_{\Omega} I((\omega, t) \in C_n) \mathsf{Q}(d\omega), \quad t \in \mathbb{R}_{++}, \ n \in \mathbb{N}.$$

Using the Fubini theorem and the property $f_n \xrightarrow[n \to \infty]{} 0$ pointwise, we get

$$\int_{\mathbb{R}_{++}} g_n(t) dt = \int_{\Omega} f_n(\omega) \mathsf{Q}(d\omega) \xrightarrow[n \to \infty]{} 0.$$

This, combined with the inequality

$$g_n(t) \geq \int_{\Omega} I((\omega, t) \in C) \mathsf{Q}(d\omega), \quad t \in \mathbb{R}_{++}, \ n \in \mathbb{N},$$

yields

$$\int_{\Omega} I((\omega, t) \in C) \mathsf{Q}(d\omega) = 0 \quad \text{for } \mu_L\text{-a.e. } t \in \mathbb{R}_{++},$$

where μ_L denotes the Lebesgue measure. Hence, $X_t = 1$ Q-a.s. for μ_L-a.e. t. The contradiction obtained shows that there exists no risk-neutral measure. □

Definition 5.20. A *combination* of arbitrage pricing models $(\Omega, \mathcal{F}, A_\gamma)$, $\gamma \in \Gamma$ is the model $\left(\Omega, \mathcal{F}, \sum_{\gamma \in \Gamma} A_\gamma\right)$, where

$$\sum_{\gamma \in \Gamma} A_\gamma := \left\{ \sum_{n=1}^{N} X_n : N \in \mathbb{N}, \ X_n \in A_{\gamma_n}, \ \gamma_n \in \Gamma \right\}.$$

5.4 Pricing of Contingent Claims

Let $(\Omega, \mathcal{F}, A)$ be an arbitrage pricing model.

Definition 5.21. A *contingent claim* is a real-valued $\mathcal{F}$-measurable function F.

Definition 5.22. **(i)** A real number x is a *fair price* of F if the combination $(\Omega, \mathcal{F}, A + A(x))$, where

$$A(x) = \{h(F - x) : h \in \mathbb{R}\},$$

satisfies the NGA condition. The set of fair prices of F will be denoted by $I(F)$.

(ii) A pair of real numbers (x, y) is a *fair bid-ask price* of F if the combination $(\Omega, \mathcal{F}, A + A(x, y))$, where

$$A(x, y) = \{g(F - y) + h(x - F) : g, h \in \mathbb{R}_+\},$$

satisfies the NGA condition. The set of fair bid-ask prices of F will be denoted by $J(F)$.

(iii) The *lower* and *upper* prices of F are defined by

$$V_*(F) = \inf\{x : x \in I(F)\},$$
$$V^*(F) = \sup\{x : x \in I(F)\}.$$

Theorem 5.23 (Main theorem for pricing contingent claims). *Suppose that the model $(\Omega, \mathcal{F}, A)$ satisfies Assumption 5.16 and the NGA condition, while F is bounded below and $\mathsf{E_Q}F < \infty$ for any $\mathsf{Q} \in \mathcal{R}$. Then*

$$I(F) \approx \Bigl[\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q}F,\ \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q}F\Bigr], \tag{5.9}$$

$$J(F) \approx \Bigl\{(x,y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q}F,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q}F\Bigr\}, \tag{5.10}$$

$$V_*(F) = \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q}F, \tag{5.11}$$

$$V^*(F) = \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q}F. \tag{5.12}$$

Proof. Equalities (5.11) and (5.12) follow from (5.9). Thus, we should only prove (5.9) (Steps 1–3 below) and (5.10) (Steps 4, 5 below).

Step 1. Let us prove the inclusion

$$I(F) \subseteq \Bigl[\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q}F,\ \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E_Q}F\Bigr].$$

Let $x \in I(F)$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Set $Z_1 = Z_0 + (F - x)$. Then $Z_1 \in B'$, where B' is defined by (5.7) with A replaced by

$$A' = \{X + h(F - x) : X \in A,\ h \in \mathbb{R}\}. \tag{5.13}$$

Set $W \equiv 1$. Lemma 5.18 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A_4'(Z_1)$ ($A_4'(Z_1)$ is defined by (5.8) with A replaced by A') and to the point W yields a probability measure Q_0 on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}_0}X \le 0$ for any $X \in A_4'(Z_1)$. By the Fatou lemma, for any $X \in A'$ such that $\frac{X}{Z_1+\gamma(Z_1)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0}\frac{X}{Z_1+\gamma(Z_1)} \le 0$. Consider the measure $\mathsf{Q} = \frac{c}{Z_1+\gamma(Z_1)}\mathsf{Q}_0$, where c is the normalizing constant. Then $\mathsf{Q} \in \mathcal{R}(Z_1) \subseteq \mathcal{R}(Z_0) = \mathcal{R}$. Moreover, $\mathsf{E_Q}(x - F) \le 0$ and $\mathsf{E_Q}(F - x) \le 0$ since the functions $\frac{x-F}{Z_1+\gamma(Z_1)}$ and $\frac{F-x}{Z_1+\gamma(Z_1)}$ are bounded below. Thus, $\mathsf{E_Q}F = x$.

Step 2. Suppose that $\mathsf{E_Q}F = \mathsf{E_{Q'}}F$ for any $\mathsf{Q}, \mathsf{Q}' \in \mathcal{R}$. Let us prove the inclusion $\mathsf{E_Q}F \in I(F)$. Denote $\mathsf{E_Q}F$ by x. Suppose that $x \notin I(F)$, i.e. the model $(\Omega, \mathcal{F}, A')$, where A' is given by (5.13), does not satisfy the NGA condition. Then there exist $Z \in B'$ and $W \in A_4'(Z) \cap (L_+^0 \setminus \{0\})$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Lemma 5.18 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A_4(Z_0)$ and to the point W yields a probability measure Q_0 on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}_0}X \le 0$ for any $X \in A_4(Z_0)$ and $\mathsf{E}_{\mathsf{Q}_0}W > 0$. Consider the measure $\mathsf{Q} = \frac{c}{Z_0+\gamma(Z_0)}\mathsf{Q}_0$, where c is the normalizing constant. Then $\mathsf{Q} \in \mathcal{R}(Z_0) = \mathcal{R}$ and $\mathsf{E_Q}W > 0$. Moreover, $\mathsf{E_Q}F = x$.

Choose an arbitrary $Y = X + h(F - x) \in A'$ (here $X \in A$) such that Y is bounded below. It follows from the condition $\mathsf{E_Q}F = x$ that $\mathsf{E_Q}X^- < \infty$. As $\mathsf{Q} \in \mathcal{R}$, we have $\mathsf{E_Q}X \le 0$. This, combined with the condition $\mathsf{E_Q}F = x$, implies that $\mathsf{E_Q}Y \le 0$. By the

Fatou lemma, Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z+\gamma(Z))\mathsf{Q}$, where c is the normalizing constant. For any $Y = X + h(F-x) \in A'$ (here $X \in A$) such that $\frac{Y}{Z+\gamma(Z)}$ is bounded below by some constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E}_\mathsf{Q} Y^- \le \mathsf{E}_\mathsf{Q}(\alpha Z + \alpha\gamma(Z)) < \infty.$$

Consequently, $\mathsf{E}_\mathsf{Q} X^- < \infty$, $\mathsf{E}_\mathsf{Q} X \le 0$, and $\mathsf{E}_\mathsf{Q} Y \le 0$. This means that $\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{Y}{Z+\gamma(Z)} \le 0$. Hence, $\mathsf{E}_{\widetilde{\mathsf{Q}}} W \le 0$. But this is a contradiction since $\widetilde{\mathsf{Q}} \sim \mathsf{Q}$ and $\mathsf{E}_\mathsf{Q} W > 0$. As a result, $x \in I(F)$.

Step 3. Let us prove the inclusion

$$\Bigl(\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F,\ \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F\Bigr) \subseteq I(F).$$

Let x belong to the left-hand side of this inclusion, i.e.

$$\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F < x < \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F.$$

Suppose that $x \notin I(F)$, i.e. the model $(\Omega, \mathcal{F}, A')$, where A' is defined by (5.13), does not satisfy the NGA condition. Then there exist $Z \in B'$ and $W \in A'_4(Z) \cap (L^0_+ \setminus \{0\})$. Applying the same reasoning as in the previous step, we find a measure $\mathsf{Q}_1 \in \mathcal{R}$ such that $\mathsf{E}_{\mathsf{Q}_1} W > 0$. By the conditions of the theorem, $\mathsf{E}_{\mathsf{Q}_1}|F| < \infty$. Find measures $\mathsf{Q}_2, \mathsf{Q}_3 \in \mathcal{R}$ such that $\mathsf{E}_{\mathsf{Q}_2} F < x$, $\mathsf{E}_{\mathsf{Q}_3} F > x$. Clearly, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_{++}$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\mathsf{E}_\mathsf{Q} F = x$, where $\mathsf{Q} = \alpha_1\mathsf{Q}_1 + \alpha_2\mathsf{Q}_2 + \alpha_3\mathsf{Q}_3$. Note that $\mathsf{Q} \in \mathcal{R}$ due to the convexity of $\mathcal{R}$ and $\mathsf{E}_\mathsf{Q} W > 0$. The proof is now completed in the same way as in the previous step.

Step 4. The arguments similar to those used in Step 1 show that

$$J(F) \subseteq \Bigl\{(x,y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F\Bigr\}.$$

Step 5. Let us prove the inclusion

$$\Bigl\{(x,y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F\Bigr\}^\circ \subseteq J(F),$$

where "$\circ$" denotes the interior. Let (x,y) belong to the left-hand side of this inclusion, i.e.

$$x < y, \qquad x < \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F, \qquad y > \inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F.$$

Suppose that $(x,y) \notin J(F)$, i.e. the model $(\Omega, \mathcal{F}, A')$, where

$$A' = \{X + g(F-y) + h(x-F) : X \in A,\ g, h \in \mathbb{R}_+\},$$

does not satisfy the NGA condition. Then there exist $Z \in B'$ and $W \in A'_4(Z) \cap (L^0_+ \setminus \{0\})$. Applying the same reasoning as in Step 2, we find a measure $\mathsf{Q}_1 \in \mathcal{R}$ such that $\mathsf{E}_{\mathsf{Q}_1} W > 0$. By the conditions of the theorem, $\mathsf{E}_{\mathsf{Q}_1}|F| < \infty$. Find measures $\mathsf{Q}_2, \mathsf{Q}_3 \in \mathcal{R}$ such that $\mathsf{E}_{\mathsf{Q}_2} F > x$, $\mathsf{E}_{\mathsf{Q}_3} F < y$. Clearly, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_{++}$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $x < \mathsf{E}_\mathsf{Q} F < y$, where $\mathsf{Q} = \alpha_1\mathsf{Q}_1 + \alpha_2\mathsf{Q}_2 + \alpha_3\mathsf{Q}_3$. Note that $\mathsf{Q} \in \mathcal{R}$ due to the convexity of $\mathcal{R}$ and $\mathsf{E}_\mathsf{Q} W > 0$. The proof is now completed in the same way as in Step 2. □

The following example shows that the equality $I(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{R}\}$ (which is true in the probability setting; see (2.9)) can be violated.

Example 5.24. Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $A = \{0\}$. Consider $F(\omega) = \omega$. Then $I(F) = (0,1)$, while $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\} = [0,1]$. □

The next example shows that the assumption "$\mathsf{E}_{\mathsf{Q}} F < \infty$ for any $\mathsf{Q} \in \mathcal{R}$" in Theorem 5.23 is essential.

Example 5.25. Let $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, and

$$A = \left\{ \sum_{n=1}^{N} h_n X_{a_n b_n} : N \in \mathbb{N},\ a_n < b_n \in \mathbb{R}_+,\ h_n \in \mathbb{R} \right\},$$

where

$$X_{ab}(\omega) = I(a < \omega \le b) - \int_a^b e^{-x} dx, \quad \omega \in \Omega.$$

Consider $F(\omega) = e^{\omega}$.

If $\mathsf{Q} \in \mathcal{R}$, then, for any $a > 0$, we have $\mathsf{E}_{\mathsf{Q}} X_{0a} = 0$ (note that X_{0a} is bounded), which means that

$$\mathsf{Q}((0,a]) = \mathsf{Q}(\mathbb{R}_{++}) \int_0^a e^{-x} dx, \quad a \in \mathbb{R}_{++}.$$

Hence, Q has the form $\alpha_1 \mathsf{Q}_1 + \alpha_2 \mathsf{Q}_2$, where $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_1 + \alpha_2 = 1$, $\mathsf{Q}_1 = \delta_0$, and Q_2 is the exponential distribution on $\mathbb{R}_+$ with parameter 1. Clearly, any measure of this form belongs to $\mathcal{R}$. We have

$$\mathsf{E}_{\mathsf{Q}} F = \begin{cases} 1 & \text{if } \mathsf{Q} = \mathsf{Q}_1, \\ +\infty & \text{otherwise.} \end{cases}$$

Consequently, $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\} = \{1\}$.

Take now $x \in I(F)$. For $n \in \mathbb{N}$, set

$$F_n(\omega) = \begin{cases} 0 & \text{if } \omega = 0, \\ e^m & \text{if } \omega \in (m, m+1],\ m = 0, \dots, n-1, \\ 0 & \text{if } \omega > n, \end{cases}$$

$$x_n = \int_0^\infty F_n(x) e^{-x} dx.$$

Then $F_n - x_n \in A$. As $x_n \to \infty$, there exists n_0 such that $x_{n_0} > x$. Then

$$(F(\omega) - x) - (F_{n_0}(\omega) - x_{n_0}) \ge x_{n_0} - x > 0, \quad \omega \in \Omega.$$

But

$$(F - x) - (F_n - x_n) \in A' = \{X + h(F - x) : X \in A,\ h \in \mathbb{R}\}.$$

This contradicts the choice of x. As a result, $I(F) = \emptyset$. □

`Remark`. Another way to define the lower and upper prices is through the sub- and superreplication, i.e.

$$C_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \le F \text{ pointwise}\}, \tag{5.14}$$

$$C^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \ge F \text{ pointwise}\}. \tag{5.15}$$

Obviously, under the assumptions of Theorem 5.23, we have

$$C_*(F) \le V_*(F) \le V^*(F) \le C^*(F).$$

In some models (for example, in the one-period model), we have $C_*(F) = V_*(F)$, $C^*(F) = V^*(F)$. However, in other models (even in the multiperiod model), these equalities might be violated (see Example 5.32).

5.5 Pricing of Controlled Contingent Claims

Let $(\Omega, \mathcal{F}, A)$ be an arbitrage pricing model.

Definition 5.26. A *controlled contingent claim* is a collection $(F_\lambda)_{\lambda\in\Lambda}$ of real-valued $\mathcal{F}$-measurable functions.

Definition 5.27. (i) A real number x is a *fair price* of $(F_\lambda)_{\lambda\in\Lambda}$ if there exists $\lambda_0 \in \Lambda$ such that the combination $(\Omega, \mathcal{F}, A + A(x, \lambda_0))$, where

$$A(x,\lambda_0) = \left\{\sum_{n=1}^{N} h_n(F_{\lambda_n} - x) + h_0(x - F_{\lambda_0}) : N \in \mathbb{N},\ \lambda_n \in \Lambda,\ h_n \in \mathbb{R}_+,\ h_0 \in \mathbb{R}_+\right\},$$

satisfies the NGA condition. The set of fair prices of $(F_\lambda)_{\lambda\in\Lambda}$ will be denoted by $I(F_\lambda; \lambda \in \Lambda)$.

(ii) A pair of real numbers (x, y) is a *fair bid-ask price* of $(F_\lambda; \lambda \in \Lambda)$ if there exists $\lambda_0 \in \Lambda$ such that the combination $(\Omega, \mathcal{F}, A + A(x, y, \lambda_0))$, where

$$A(x,y,\lambda_0) = \left\{\sum_{n=1}^{N} h_n(F_{\lambda_n} - y) + h_0(x - F_{\lambda_0}) : N \in \mathbb{N},\ \lambda_n \in \Lambda,\ h_n \in \mathbb{R}_+,\ h_0 \in \mathbb{R}_+\right\},$$

satisfies the NGA condition. The set of fair bid-ask prices of $(F_\lambda)_{\lambda\in\Lambda}$ will be denoted by $J(F_\lambda; \lambda \in \Lambda)$.

(iii) The *lower* and *upper* prices of $(F_\lambda)_{\lambda\in\Lambda}$ are defined by

$$V_*(F_\lambda; \lambda \in \Lambda) = \inf\{y : (x, y) \in J(F_\lambda; \lambda \in \Lambda)\},$$
$$V^*(F_\lambda; \lambda \in \Lambda) = \sup\{x : (x, y) \in J(F_\lambda; \lambda \in \Lambda)\}.$$

Theorem 5.28 (Main theorem for pricing controlled contingent claims). *Suppose that the model $(\Omega, \mathcal{F}, A)$ satisfies Assumption 5.16 and the NGA condition, while F_λ is bounded below for any $\lambda \in \Lambda$ and $\sup_{\lambda\in\Lambda} \mathsf{E}_\mathsf{Q} F_\lambda < \infty$ for any $\mathsf{Q} \in \mathcal{R}$. Then*

$$I(F_\lambda; \lambda \in \Lambda) \subseteq \left[\inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_\mathsf{Q} F_\lambda,\ \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_\mathsf{Q} F_\lambda\right], \tag{5.16}$$

$$J(F_\lambda; \lambda \in \Lambda) \approx \left\{(x, y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_\mathsf{Q} F_\lambda,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_\mathsf{Q} F_\lambda\right\}, \tag{5.17}$$

$$V_*(F_\lambda; \lambda \in \Lambda) = \inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_\mathsf{Q} F_\lambda, \tag{5.18}$$

$$V^*(F_\lambda; \lambda \in \Lambda) = \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_\mathsf{Q} F_\lambda. \tag{5.19}$$

`Proof`. We will check only (5.17). Inclusion (5.16) is verified similarly, while equalities (5.18) and (5.19) follow from (5.17).

Step 1. Let us prove the inclusion

$$J(F_\lambda; \lambda \in \Lambda) \subseteq \left\{(x, y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_\mathsf{Q} F_\lambda,\ y \ge \inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E}_\mathsf{Q} F_\lambda\right\}. \tag{5.20}$$

Let $(x, y) \in J(F_\lambda; \lambda \in \Lambda)$. Let λ_0 be an element of Λ such that the model $(\Omega, \mathcal{F}, A + A(x, y, \lambda_0))$ satisfies the NGA condition. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Set

$Z_1 = Z_0 + (F_{\lambda_0} - y)$. Employing the same reasoning as in the proof of Theorem 5.23 (Step 1), we find a measure $\mathsf{Q} \in \mathcal{R}$ such that $\mathsf{E_Q}(x - F_{\lambda_0}) \le 0$ and $\mathsf{E_Q}(F_\lambda - y) \le 0$ for any $\lambda \in \Lambda$. Then

$$x \le \mathsf{E_Q} F_{\lambda_0} \le \sup_{\lambda \in \Lambda} \mathsf{E_Q} F_\lambda \le y,$$

which means that (x, y) belongs to the right-hand side of (5.20).

Step 2. Let us prove the inclusion

$$\Big\{(x,y) : x \le y,\ x \le \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda,\ y \ge \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda\Big\}^\circ \subseteq J(F_\lambda; \lambda \in \Lambda),$$

where "$\circ$" denotes the interior. Let (x, y) belong to the left-hand side of this inclusion, i.e.

$$x < y, \qquad x < \sup_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda, \qquad y > \inf_{\mathsf{Q}\in\mathcal{R}} \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda.$$

Find measures $\mathsf{Q}_1, \mathsf{Q}_2 \in \mathcal{R}$ such that

$$\sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}_1} F_\lambda > x, \qquad \sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}_2} F_\lambda < y.$$

Set $\mathsf{Q}(\alpha) = (1-\alpha)\mathsf{Q}_1 + \alpha \mathsf{Q}_2$, $\alpha \in [0,1]$. As the map $\alpha \mapsto \sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}(\alpha)} F_\lambda$ is continuous in α, there exists $\alpha_0 \in (0,1)$ such that

$$x < \sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}(\alpha_0)} F_\lambda < y.$$

Note that $\mathsf{Q}(\alpha_0) \in \mathcal{R}$ due to the convexity of $\mathcal{R}$. Find $\lambda_0 \in \Lambda$ such that $\mathsf{E}_{\mathsf{Q}(\alpha_0)} F_{\lambda_0} > x$. Suppose that the model $(\Omega, \mathcal{F}, A')$, where

$$A' = \Big\{X + \sum_{n=1}^N h_n(F_{\lambda_n} - y) + h_0(x - F_{\lambda_0}) : X \in A,\ N \in \mathbb{N},\ \lambda_n \in \Lambda,\ h_n \in \mathbb{R}_+,\ h_0 \in \mathbb{R}_+\Big\},$$

does not satisfy the NGA condition. Then there exist $Z \in B'$ and $W \in A_4'(Z) \cap (L_+^0 \setminus \{0\})$ ($A_4'(Z)$ is defined by (5.8) with A replaced by A'). Applying the same reasoning as in the proof of Theorem 5.23 (Step 2), we find a measure $\mathsf{Q}_3 \in \mathcal{R}$ such that $\mathsf{E}_{\mathsf{Q}_3} W > 0$. By the conditions of the theorem, $\sup_{\lambda\in\Lambda} \mathsf{E}_{\mathsf{Q}_3} F_\lambda < \infty$. Clearly, there exists $\varepsilon \in (0,1)$ such that

$$x < \mathsf{E_Q} F_{\lambda_0} \le \sup_{\lambda\in\Lambda} \mathsf{E_Q} F_\lambda < y,$$

where $\mathsf{Q} = (1-\varepsilon)\mathsf{Q}(\alpha_0) + \varepsilon \mathsf{Q}_3$. Then $\mathsf{Q} \in \mathcal{R}$ due to the convexity of $\mathcal{R}$ and $\mathsf{E_Q} W > 0$. The proof is now completed in the same way as in Theorem 5.23 (Step 2). □

5.6 Particular Models

The main results for each of the models considered in Chapters 3 and 4 are:

1. the proof that Assumption 2.11 is satisfied;
2. the description of $\mathcal{R}$.

These results can easily be transferred to the possibility framework. (Of course, the results that employ some particular probabilistic structure, like Examples 3.13, 3.14, 3.15, Proposition 3.20, and Corollary 3.21, cannot be transferred to the possibility setting.) In order to do this, one should

1. Replace the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ (resp., the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P})$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$, or $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathsf{P})$) by the possibility space $(\Omega, \mathcal{F})$ (resp., the *filtered possibility space* $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N})$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]})$, or $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$).
2. Replace the condition "$\mathcal{F}_0$ is P-trivial" by the condition "$\mathcal{F}_0 = \{\emptyset, \Omega\}$".
3. Replace the condition "S_1 is an $\mathbb{R}^d_+$-valued random variable on $(\Omega, \mathcal{F}, \mathsf{P})$" (resp., "$(S_n)_{n=0,\dots,N}$ is an $\mathbb{R}^d_+$-valued $(\mathcal{F}_n)$-adapted random sequence" or "$(S_t)_{t\in[0,T]}$ is an $\mathbb{R}^d_+$-valued $(\mathcal{F}_t)$-adapted càdlàg process", etc.) by the condition "S_1 is an $\mathbb{R}^d_+$-valued $\mathcal{F}$-measurable function" (resp., "$(S_n)_{n=0,\dots,N}$ is a collection of $\mathbb{R}^d_+$-valued $(\mathcal{F}_n)$-measurable functions" or "$(S_t)_{t\in[0,T]}$ is a collection of $\mathbb{R}^d_+$-valued $(\mathcal{F}_t)$-measurable functions such that, for any $\omega \in \Omega$, the map $t \mapsto S_t(\omega)$ is càdlàg", etc.).
4. Replace the condition "$\mathsf{Q} \sim \mathsf{P}$" in the definition of $\mathcal{M}$ by the condition "Q is a probability measure on $\mathcal{F}$".

Also note that in the probability approach all the random variables are considered as the classes of equivalence under the indistinguishability relation, while in the possibility approach we consider measurable functions with no nontrivial equivalence relation.

Then the Key Lemmas of Sections 3.1–4.4 (together with their proofs) are automatically transferred to the possibility framework and

- Theorem 5.17 yields the necessary and sufficient conditions for the absence of the generalized arbitrage;
- Theorem 5.23 yields the form of fair prices of a contingent claim (that satisfies the conditions of this theorem);
- Theorem 5.28 yields the form of fair prices of a controlled contingent claim (that satisfies the conditions of this theorem).

5.7 Multiperiod Model

We will first consider the frictionless model, i.e. the model of Section 3.2 with the changes described in Section 5.6.

Recall that an *atom* of a σ-field $\mathcal{F}$ is a set $\mathfrak{a} \in \mathcal{F}$ such that $\mathfrak{a} \neq \emptyset$ and, for any $D \in \mathcal{F}$, we have either $D \supseteq \mathfrak{a}$ or $D \cap \mathfrak{a} = \emptyset$.

`Notation`. Suppose that, for any $n = 0, \dots, N-1$, $\omega \in \Omega$, there exists an atom $\mathfrak{a}_n(\omega)$ of $\mathcal{F}_n$ that contains ω. Set $C_n(\omega) = \overline{\operatorname{conv}}\{\overline{S}_{n+1}(\omega') : \omega' \in \mathfrak{a}_n(\omega)\}$ and let $C_n^\circ(\omega)$ denote the relative interior of $C_n(\omega)$.

Theorem 5.29 (FTAP). *Suppose that, for any $n = 0, \dots, N-1$, $\omega \in \Omega$, there exists an atom $\mathfrak{a}_n(\omega)$ of $\mathcal{F}_n$ that contains ω. Then, for the model $(\Omega, \mathcal{F}, A)$, the following conditions are equivalent:*

(a) *NGA;*
(b) *NA (i.e. $A \cap L^0_+ = \{0\}$);*
(c) $\overline{S}_n(\omega) \in C_n^\circ(\omega)$, $n = 0, \dots, N-1$, $\omega \in \Omega$;
(d) *for any $D \in \mathcal{F} \setminus \{\emptyset\}$, there exists $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(D) > 0$ ($\mathcal{M}$ denotes the set of martingale measures for $\overline{S}$).*

`Proof`. *Step 1.* The implication (a)$\Rightarrow$(b) is obvious.

Step 2. Let us prove the implication (b)$\Rightarrow$(c). Suppose that there exist $m \in \{0,\dots,N-1\}$ and $\omega_0 \in \Omega$ such that $\overline{S}_m(\omega_0) \notin C_m^\circ(\omega_0)$. By the separation theorem, there exists $h \in \mathbb{R}^d$ such that $\langle h, (\overline{S}_{m+1}(\omega) - \overline{S}_m(\omega))\rangle \geq 0$ for any $\omega \in \mathfrak{a}_m(\omega_0)$ and $\langle h, (\overline{S}_{m+1}(\omega) - \overline{S}_m(\omega))\rangle > 0$ for some $\omega \in \mathfrak{a}_m(\omega_0)$. Set

$$H_n(\omega) = \begin{cases} hI(\omega \in \mathfrak{a}_m(\omega_0)) & \text{if } n = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=1}^{N}\sum_{i=1}^{d} H_n^i(\overline{S}_n^i - \overline{S}_{n-1}^i) \in A \cap (L_+^0 \setminus \{0\}),$$

which contradicts the NA condition.

Step 3. Let us prove the implication (c)$\Rightarrow$(d). Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Choose $\omega_0 \in D$. Take $\omega_1,\dots,\omega_m \in \Omega$ and $\alpha_0,\dots,\alpha_m \in \mathbb{R}_{++}$ provided by Lemma 5.30. Then the measure $\mathsf{Q} = \sum_{k=0}^m \alpha_k \delta_{\omega_k}$ belongs to $\mathcal{M}$ and $\mathsf{Q}(D) > 0$.

Step 4. The implication (d)$\Rightarrow$(a) follows from Theorem 5.17 and the Key Lemma for the model under consideration (it states that $\mathcal{R} = \mathcal{R}\big(\sum_{n=1}^N \sum_{i=1}^d (\overline{S}_n^i - \overline{S}_0^i)\big) = \mathcal{M}$). □

Lemma 5.30. *Suppose that condition* (c) *of Theorem 5.29 is satisfied. Let* $\omega_0 \in \Omega$. *Then there exist* $\omega_1,\dots,\omega_m \in \Omega$ *and* $\alpha_0,\dots,\alpha_m \in \mathbb{R}_{++}$ *such that* $\sum_{k=0}^m \alpha_k = 1$ *and* $\sum_{k=0}^m \alpha_k X(\omega_k) = 0$ *for any* $X \in A$.

`Proof.` We will prove this statement by the induction in N.

Base of induction. For $N = 1$, the statement is verified by the same arguments as those used in the proof of Theorem 5.9 (Step 2).

Step of induction. Assume that the statement is true for $N-1$. Let us prove it for N. By the induction hypothesis, there exist $\widetilde{\omega}_1,\dots,\widetilde{\omega}_l \in \Omega$ and $\widetilde{\alpha}_0,\dots,\widetilde{\alpha}_l \in \mathbb{R}_{++}$ such that $\widetilde{\omega}_0 = \omega_0$, $\sum_{i=0}^l \widetilde{\alpha}_i = 1$, and $\sum_{i=0}^l \widetilde{\alpha}_i X(\widetilde{\omega}_i) = 0$ for any $X \in A'$, where

$$A' = \left\{\sum_{n=1}^{N-1}\sum_{i=1}^{d} H_n^i(\overline{S}_n^i - \overline{S}_{n-1}^i) : H_n^i \text{ is } \mathcal{F}_{n-1}\text{-measurable}\right\}.$$

For any $i = 0,\dots,l$, there exist $\widetilde{\omega}_{i0},\dots,\widetilde{\omega}_{il(i)} \in \mathfrak{a}_{N-1}(\widetilde{\omega}_i)$ and $\widetilde{\alpha}_{i0},\dots,\widetilde{\alpha}_{il(i)} \in \mathbb{R}_{++}$ such that $\widetilde{\omega}_{i0} = \widetilde{\omega}_i$, $\sum_{j=0}^{l(i)} \widetilde{\alpha}_{ij} = 1$, and

$$\sum_{j=0}^{l(i)} \widetilde{\alpha}_{ij}(\overline{S}_N(\widetilde{\omega}_{ij}) - \overline{S}_{N-1}(\widetilde{\omega}_{ij})) = 0.$$

Let $(i(0),j(0)),\dots,(i(m),j(m))$ be a numbering of the set $\{(i,j) : i = 0,\dots,l,\ j = 0,\dots,l(i)\}$. We arrange this numbering in such a way that $i(0) = j(0) = 0$. Set $\omega_k = \widetilde{\omega}_{i(k)j(k)}$, $\alpha_k = \widetilde{\alpha}_{i(k)}\widetilde{\alpha}_{i(k)j(k)}$, $k = 0,\dots,m$. Then, for any

$$X = \sum_{n=1}^{N} \langle H_n, (\overline{S}_n - \overline{S}_{n-1})\rangle \in A,$$

we have

$$\sum_{k=0}^{m} \alpha_k X(\omega_k) = \sum_{n=1}^{N-1} \sum_{i=0}^{l} \sum_{j=0}^{l(i)} \widetilde{\alpha}_i \widetilde{\alpha}_{ij} \langle H_n(\widetilde{\omega}_{ij}), (\overline{S}_n(\widetilde{\omega}_{ij}) - \overline{S}_{n-1}(\widetilde{\omega}_{ij}))\rangle$$
$$+ \sum_{i=0}^{l} \sum_{j=0}^{l(i)} \widetilde{\alpha}_i \widetilde{\alpha}_{ij} \langle H_N(\widetilde{\omega}_{ij}), (\overline{S}_N(\widetilde{\omega}_{ij}) - \overline{S}_{N-1}(\widetilde{\omega}_{ij}))\rangle$$
$$= \sum_{n=1}^{N-1} \sum_{i=0}^{l} \widetilde{\alpha}_i \langle H_n(\widetilde{\omega}_i), (\overline{S}_n(\widetilde{\omega}_i) - \overline{S}_{n-1}(\widetilde{\omega}_i))\rangle$$
$$+ \sum_{i=0}^{l} \widetilde{\alpha}_i \Big\langle H_N(\widetilde{\omega}_i), \sum_{j=0}^{l(j)} \widetilde{\alpha}_{ij} (\overline{S}_N(\widetilde{\omega}_{ij}) - \overline{S}_{N-1}(\widetilde{\omega}_{ij})) \Big\rangle = 0.$$

In the second equality, we used the fact that H_n, $n = 0, \dots, N$ and $\overline{S}_n$, $n = 0, \dots, N-1$ are constant on the atoms of $\mathcal{F}_{N-1}$. Thus, $\omega_1, \dots, \omega_m$ and $\alpha_0, \dots, \alpha_m$ satisfy the desired conditions. □

Theorem 5.29 yields

Corollary 5.31. *Suppose that* $\overline{S}_0 \in \mathbb{R}^d_{++}$,

$$\{(\overline{S}_1(\omega), \dots, \overline{S}_N(\omega)) : \omega \in \Omega\} = (\mathbb{R}^d_{++})^N,$$

$\mathcal{F}_n = \mathcal{F}^{\overline{S}}_n$, *and* $\mathcal{F} = \mathcal{F}_N$. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

The following example shows that in the model under consideration the lower and upper prices $V_*(F)$ and $V^*(F)$ provided by Definition 5.22 do not coincide with the values $C_*(F)$ and $C^*(F)$ defined by (5.14) and (5.15).

Example 5.32. Let $\Omega = \mathbb{R}^2_{++}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(X_1)$, $\mathcal{F}_2$ be the universal completion of $\mathcal{B}(\Omega)$ (i.e. $\mathcal{F}_2 = \bigcap_{\mathsf{Q}} \overline{\mathcal{B}(\Omega)}^{\mathsf{Q}}$, where $\overline{\mathcal{B}(\Omega)}^{\mathsf{Q}}$ denotes the Q-completion of $\mathcal{B}(\Omega)$, and the intersection is taken over all the probability measures Q on $\mathcal{B}(\Omega)$), $\mathcal{F} = \mathcal{F}_2$, $\overline{S}_0 = 1$, $\overline{S}_1 = 1$, $\overline{S}_2 = X_2$, where X_1, X_2 are the coordinate maps $\Omega \to \mathbb{R}_{++}$. It follows from Theorem 5.29 that this model satisfies the NGA condition. Let $D \subset \mathbb{R}_{++}$ be a set such that $D \notin \mathcal{B}(\mathbb{R}_{++})$ and D belongs to the universal completion of $\mathcal{B}(\mathbb{R}_{++})$ (as an example of such a set, one can take a Suslin set that is not Borel; see [F96; 2.2.11]). Consider $F = (X_2 - 1)I(X_1 \in D)$.

Let us prove that $C^*(F) = 1$. Since $1 + \overline{S}_2 - \overline{S}_1 \ge F$, we get $C^*(F) \le 1$. Suppose now that $C^*(F) < 1$. Then there exist $x < 1$ and an $\mathcal{F}_1$-measurable function H such that $x + H(\overline{S}_2 - \overline{S}_1) \ge F$. There exists a $\mathcal{B}(\mathbb{R}_{++})$-measurable function $\widetilde{H}$ such that $H = \widetilde{H}(X_1)$ pointwise. We have

$$x + \widetilde{H}(x_1)(x_2 - 1) \ge x_2 - 1, \quad x_1 \in D,\ x_2 \in \mathbb{R}_{++},$$
$$x + \widetilde{H}(x_1)(x_2 - 1) \ge 0, \quad x_1 \notin D,\ x_2 \in \mathbb{R}_{++}.$$

Consequently, $\widetilde{H} \ge 1$ on D and $\widetilde{H} \le x$ on $\mathbb{R}_{++} \setminus D$. This implies that $\{H > x\} = \{X_1 \in D\} \notin \mathcal{F}_1$. The contradiction obtained shows that $C^*(F) = 1$.

For any $\mathsf{Q} \in \mathcal{R}$, we have $\mathsf{E}_{\mathsf{Q}}(X_2 \,|\, \mathcal{F}_1) = \mathsf{E}_{\mathsf{Q}}(\overline{S}_2 \,|\, \mathcal{F}_1) = 1$ and $\{X_1 \in D\} \in \overline{\mathcal{F}_1}^{\mathsf{Q}}$. Hence, $\mathsf{E}_{\mathsf{Q}} F = 0$. Consequently, $V^*(F) = 0$. □

Let us now consider the multiperiod model with friction. Recall that in this model the set of attainable incomes is defined as

$$A = \Big\{ \sum_{m=1}^{M} \sum_{i=1}^{d} \big[G_m^i \big((1-\alpha^i)\overline{S}_{v_m}^i - \overline{S}_{u_m}^i \big) + H_m^i \big(-\overline{S}_{v_m}^i + (1-\alpha^i-\beta^i)\overline{S}_{u_m}^i \big) \big] : M \in \mathbb{N},$$

$$u_m \le v_m \text{ are } (\mathcal{F}_n)\text{-stopping times, } G_m^i, H_m^i \text{ are } \mathbb{R}_+\text{-valued and } \mathcal{F}_{u_m}\text{-measurable} \Big\}.$$

We present only a sufficient condition for the absence of the generalized arbitrage.

Corollary 5.33. *Suppose that* $\overline{S}_0 \in \mathbb{R}_{++}^d$,

$$\{(\overline{S}_1(\omega), \dots, \overline{S}_N(\omega)) : \omega \in \Omega\} = (\mathbb{R}_{++}^d)^N,$$

$\mathcal{F}_n = \mathcal{F}_n^{\overline{S}}$, *and* $\mathcal{F} = \mathcal{F}_N$. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

`Proof.` For any

$$X = \sum_{m=1}^{M} \sum_{i=1}^{d} \big[G_m^i \big((1-\alpha^i)\overline{S}_{v_m}^i - \overline{S}_{u_m}^i \big) + H_m^i \big(-\overline{S}_{v_m}^i + (1-\alpha^i-\beta^i)\overline{S}_{u_m}^i \big) \big] \in A,$$

we have

$$X \le \sum_{m=1}^{M} \sum_{i=1}^{d} \big[G_m^i \big(\overline{S}_{v_m}^i - \overline{S}_{u_m}^i \big) + H_m^i \big(-\overline{S}_{v_m}^i + \overline{S}_{u_m}^i \big) \big].$$

The result now follows from Corollary 5.31. □

5.8 Continuous-Time Model with a Finite Time Horizon

We will first consider the frictionless model, i.e. the model of Section 3.3 with the changes described in Section 5.8. We present two sufficient conditions for the absence of the generalized arbitrage.

Proposition 5.34. *Suppose that* $\overline{S}_0 \in \mathbb{R}_{++}^d$,

$$\{\overline{S}_\cdot(\omega) : \omega \in \Omega\} = \{f : f \text{ is a càdlàg piecewise constant function } [0,T] \to \mathbb{R}_{++}^d \text{ with a finite number of jumps, } f(0) = \overline{S}_0\},$$

$\mathcal{F}_t = \mathcal{F}_t^{\overline{S}}$, *and* $\mathcal{F} = \mathcal{F}_T$. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

`Proof.` Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. Let $0 < t_1 < \dots < t_N \le T$ be the jump times of $\overline{S}_\cdot(\omega_0)$. We set $t_0 = 0$, $t_{N+1} = T$. It follows from Lemma 5.30 that there exist $\omega_1, \dots, \omega_m \in \Omega$ and $\alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}$ such that $\overline{S}_\cdot(\omega_k)$ is constant on $[t_l, t_{l+1})$, $k = 0, \dots, m$, $l = 0, \dots, N$, $\sum_{k=0}^m \alpha_k = 1$, and the sequence $(\overline{S}_{t_0}, \dots, \overline{S}_{t_{N+1}})$ is an $(\mathcal{F}_{t_0}, \dots, \mathcal{F}_{t_{N+1}})$-martingale with respect to the measure $\mathsf{Q} = \sum_{k=0}^m \alpha_k \delta_{\omega_k}$. As $\overline{S}$ is Q-a.s. constant on $[t_l, t_{l+1})$, $l = 0, \dots, N$, the process $(\overline{S}_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. By the Key Lemma for the model under consideration (it states that $\mathcal{R} = \mathcal{R}\big(\sum_{i=1}^d (\overline{S}_T^i - \overline{S}_0^i)\big) = \mathcal{M}$), we have $\mathsf{Q} \in \mathcal{R}$. Moreover, $\mathsf{Q}(D) > 0$. An application of Theorem 5.17 completes the proof. □

Proposition 5.35. *Suppose that* $\overline{S}_0 \in \mathbb{R}^d_{++}$,

$$\{\overline{S}_\cdot(\omega) : \omega \in \Omega\} = \big\{f : f \text{ is a càdlàg function } [0,T] \to \mathbb{R}^d_{++} \text{ with finite variation such that } \inf_{t\in[0,T]} f^i(t) > 0,\ i = 1,\dots,d \text{ and } f(0) = \overline{S}_0\big\},$$

$\mathcal{F}_t = \mathcal{F}_t^{\overline{S}}$, *and* $\mathcal{F} = \mathcal{F}_T$. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

Proof. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. Set $\varphi(t) = \overline{S}_t(\omega_0)$, $\psi^i(t) = \ln \varphi^i(t)$, $i = 1,\dots,d$, $t \in [0,T]$. For each $i = 1,\dots,d$, the function ψ^i can be represented as $\psi = \psi^i_+ - \psi^i_-$, where ψ^i_+ and ψ^i_- are càdlàg and increasing. Set

$$\lambda^i_+(t) = \frac{\psi^i_-(t)}{e-1}, \quad \lambda^i_-(t) = \frac{\psi^i_+(t)}{1-e^{-1}}, \quad i = 1,\dots,d,\ t \in [0,T].$$

Let N^i_+, N^i_-, $i = 1,\dots,d$ be independent Poisson processes with intensity 1. For each $i = 1,\dots,d$, the process

$$Z^i_t = \exp\big\{(N^i_+)_{\lambda^i_+(t)} - (N^i_-)_{\lambda^i_-(t)} - (e-1)\lambda^i_+(t) + (1-e^{-1})\lambda^i_-(t)\big\}, \quad t \in [0,T]$$

is a martingale with respect to its natural filtration. Consider the space

$$\mathcal{V} = \big\{f : f \text{ is a càdlàg function } [0,T] \to \mathbb{R}^d_{++} \text{ with finite variation such that } \inf_{t\in[0,T]} f^i(t) > 0,\ i = 1,\dots,d \text{ and } f(0) = \overline{S}_0\big\}$$

equipped with the σ-field $\mathcal{G} = \sigma(X_t; t \in [0,T])$, where $X_t(f) = f(t)$. Set $\mathsf{Q}_0 = \operatorname{Law}(Z_t; t \in [0,T])$. Then X is an $(\mathcal{F}_t^X, \mathsf{Q}_0)$-martingale. In view of the representation

$$Z^i_t = \varphi^i(t) \exp\big\{(N^i_+)_{\lambda^i_+(t)} - (N^i_-)_{\lambda^i_-(t)}\big\}, \quad i = 1,\dots,d,\ t \in [0,T],$$

we have $\mathsf{Q}_0(\{\varphi\}) > 0$. Define the measure Q on $\{\overline{S}_\cdot^{-1}(C) : C \in \mathcal{G}\}$ by $\mathsf{Q}(\overline{S}_\cdot^{-1}(C)) := \mathsf{Q}_0(C)$. Note that $\{\overline{S}_\cdot^{-1}(C) : C \in \mathcal{G}\} = \mathcal{F}$ and Q is correctly defined. Then $\overline{S}$ is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. By the Key Lemma for the model under consideration, $\mathsf{Q} \in \mathcal{R}$. Moreover, the set $\overline{S}_\cdot^{-1}(\{\varphi\})$ contains ω_0 and is an atom of $\mathcal{F}$. Hence, $\overline{S}_\cdot^{-1}(\{\varphi\}) \subseteq D$, and therefore, $\mathsf{Q}(D) > 0$. An application of Theorem 5.17 completes the proof. □

The following statement is rather surprising.

Proposition 5.36. *Suppose that* $\overline{S}_0 \in \mathbb{R}^d_{++}$,

$$\{\overline{S}_\cdot(\omega) : \omega \in \Omega\} = \{f : f \text{ is a continuous function } [0,T] \to \mathbb{R}^d_{++},\ f(0) = \overline{S}_0\},$$

$\mathcal{F}_t = \mathcal{F}_t^{\overline{S}}$, *and* $\mathcal{F} = \mathcal{F}_T$. *Then the model* $(\Omega, \mathcal{F}, A)$ *does not satisfy the NGA condition.*

Proof. Suppose that the NGA condition is satisfied. By Theorem 5.17, there exists a measure $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(D) > 0$, where $D = \big\{\overline{S}^1_t = 1 + t,\ t \in [0,T]\big\}$. By the Key Lemma for the model under consideration, $\overline{S}$ is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. Moreover, $\overline{S}$ is continuous. On the set D, the quadratic variation of $\overline{S}^1$ is 0. This implies that $\overline{S}^1_T = \overline{S}^1_0$ Q-a.e. on D (see [RY99; Ch. IV, Prop. 1.13]). The obtained contradiction shows that the NGA condition is not satisfied. □

Let us now consider the model with friction, i.e. the model of Section 4.2 with the changes described in Section 5.6. For this model, we are able to prove the absence of the generalized arbitrage under more natural assumptions than those used for the frictionless model.

Proposition 5.37. *Suppose that* $\overline{S}_0 \in \mathbb{R}^d_{++}$,

$$\{\overline{S}_{\cdot}(\omega) : \omega \in \Omega\} = \big\{f : f \text{ is a càdlàg function } [0,T] \to \mathbb{R}^d_{++} \text{ such that } \inf_{t\in[0,T]} f^i(t) > 0,\ i = 1,\dots,d \text{ and } f(0) = \overline{S}_0\big\}, \tag{5.21}$$

$\mathcal{F}_t = \mathcal{F}_t^{\overline{S}}$, *and* $\mathcal{F} = \mathcal{F}_T$. *Suppose moreover that* $\alpha^i > 0$, $i = 1,\dots,d$. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

Proof. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. Consider the function $\varphi(t) = \overline{S}_t(\omega_0)$, $t \in [0,T]$. There exists a càdlàg function $\psi : [0,T] \to \mathbb{R}^d_{++}$ with the properties: ψ is piecewise constant with a finite number of jumps, $\psi(0) = \overline{S}_0$, and

$$(1-\alpha^i)\varphi^i(t) \le \psi^i(t) \le \varphi^i(t), \quad i = 1,\dots,d,\ t \in [0,T]. \tag{5.22}$$

Take $\omega_0' \in \Omega$ such that $\overline{S}_t(\omega_0') = \psi(t)$. The reasoning used in the proof of Proposition 5.34 shows that there exist $\omega_1',\dots,\omega_m' \in \Omega$ and $\alpha_0,\dots,\alpha_m \in \mathbb{R}_{++}$ such that $\sum_{k=0}^m \alpha_k = 1$ and

$$\sum_{k=0}^m \sum_{i=1}^d \alpha_k H^i(\omega_k')\big(\overline{S}^i_u(\omega_k') - \overline{S}^i_v(\omega_k')\big) = 0 \tag{5.23}$$

for any $(\mathcal{F}_t)$-stopping times $u \le v$ and any $\mathbb{R}^d$-valued $\mathcal{F}_u$-measurable function H. Set $\omega_k = \omega_k'$, $k = 1,\dots,m$. Consider an arbitrary element

$$X = \sum_{n=1}^N \sum_{i=1}^d \big[G^i_n\big((1-\alpha^i)\overline{S}^i_{v_n} - \overline{S}^i_{u_n}\big) + H^i_n\big(-\overline{S}^i_{v_n} + (1-\alpha^i-\beta^i)\overline{S}^i_{u_n}\big)\big] \in A.$$

Set

$$Y = \sum_{n=1}^N \sum_{i=1}^d \big[G^i_n\big(\overline{S}^i_{v_n} - \overline{S}^i_{u_n}\big) + H^i_n\big(-\overline{S}^i_{v_n} + \overline{S}^i_{u_n}\big)\big].$$

Then

$$\sum_{k=0}^m \alpha_k X(\omega_k) \le \alpha_0 Y(\omega_0') + \sum_{k=1}^m \alpha_k X(\omega_k) \le \alpha_0 Y(\omega_0') + \sum_{k=1}^m \alpha_k Y(\omega_k) = \sum_{k=0}^m \alpha_k Y(\omega_k')$$

(in the first inequality, we applied (5.22)). It follows from (5.23) that $\sum_{k=0}^m \alpha_k Y(\omega_k') = 0$. Hence, the measure $\mathsf{Q} = \sum_{k=0}^m \alpha_k \delta_{\omega_k}$ belongs to $\mathcal{R}$. Moreover, $\mathsf{Q}(D) > 0$. An application of Theorem 5.17 completes the proof. □

Remark. Proposition 5.37 remains true if we replace the word "càdlàg" in (5.21) by the word "continuous". The proof is the same.

5.9 Continuous-Time Model with the Infinite Time Horizon

Let us consider the model of Section 3.4 with the changes described in Section 5.6. In this section, we will show that the additional reasoning of Section 3.4 related to the existence of $\lim_{t\to\infty} \overline{S}_t$ can be transferred to the possibility setting.

Key Lemma 5.38. *Suppose that the limit* $\lim_{t\to\infty} \overline{S}_t(\omega)$ *exists for any* $\omega \in \Omega$. *Then, for the model* $(\Omega, \mathcal{F}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}\biggl(\sum_{i=1}^{d}\bigl(\overline{S}^i_\infty - \overline{S}^i_0\bigr)\biggr) = \mathcal{M}$$

($\mathcal{M}$ denotes the set of uniformly integrable martingale measures for $\overline{S}$).

The proof is similar to the proof of Key Lemma 3.16.

Corollary 5.39. *The model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition if and only if for any* $D \in \mathcal{F} \setminus \{\emptyset\}$, *there exists a uniformly integrable martingale measure* Q *such that* $\mathsf{Q}(D) > 0$.

`Proof`. *Step 1.* Let us prove the "only if" implication. Suppose that the set

$$C = \Bigl\{\omega \in \Omega : \lim_{t\to\infty} \overline{S}_t(\omega) \text{ does not exist}\Bigr\} \tag{5.24}$$

is nonempty. Note that $C \in \mathcal{F}$ in view of the equality

$$C = \bigcup_{i=1}^{d}\Bigl\{\omega \in \Omega : \liminf_{t\to\infty} \overline{S}^i_t(\omega) < \limsup_{t\to\infty} \overline{S}^i_t(\omega)\Bigr\}.$$

Consider $W = I_C$. Lemma 5.18 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A_4(0)$ and to the point W yields a probability measure Q on $\mathcal{F}$ such that $\mathsf{Q}(C) > 0$ and $\mathsf{E}_\mathsf{Q} X \le 0$ for any $X \in A$ that is bounded below. For any $i = 1, \dots, d$, any $u \le v \in \mathbb{R}_+$, and any $D \in \mathcal{F}_u$ such that $\overline{S}^i_u$ is bounded on D, the random variable $I_D\bigl(\overline{S}^i_v - \overline{S}^i_u\bigr)$ is bounded below, and hence, $\mathsf{E}_\mathsf{Q} I_D\bigl(\overline{S}^i_v - \overline{S}^i_u\bigr) \le 0$. This shows that $\overline{S}^i$ is an $(\mathcal{F}_t, \mathsf{Q})$-supermartingale. By Doob's supermartingale convergence theorem (see [RY99; Ch. II, Th. 2.10]), the limit $\lim_{t\to\infty} \overline{S}^i_t$ exists Q-a.s. The obtained contradiction shows that $C = \emptyset$. Now, Theorem 5.17, combined with Key Lemma 5.38, yields the desired statement.

Step 2. Let us prove the "if" implication. Suppose that the set C defined by (5.24) is nonempty. Take $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(C) > 0$. By Doob's theorem, $\lim_{t\to\infty} \overline{S}_t$ exists Q-a.s. The obtained contradiction shows that $C = \emptyset$. Now, Theorem 5.17, combined with Key Lemma 5.38, yields the desired statement. □

It has been shown in the proof of Corollary 5.39 that the NGA condition implies the existence of $\lim_{t\to\infty} \overline{S}_t(\omega)$ for each $\omega \in \Omega$. Hence, Theorems 5.23 and 5.28 can be applied with no additional assumptions.

5.10 Model with European Options

We will first consider the frictionless model, i.e. the model of Section 3.6 with the changes described in Section 5.6. This model will be studied in two special cases:

1. the case, where $\mathbb{K}^i = \mathbb{R}_+$, $i = 1, \dots, d$;
2. the case, where $\mathbb{K}^i$ is finite, $i = 1, \dots, d$.

Propositions 5.40 and 5.42 show that in case 1 the NGA condition is not satisfied in most natural situations, while in case 2 the NGA condition is satisfied in most natural situations.

Proposition 5.40. *Suppose that* $\mathbb{K}^i = \mathbb{R}_+$, $i = 1, \dots, d$. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition if and only if the set* $\{S_T(\omega) : \omega \in \Omega\}$ *is countable and, for any* $a \in \{S_T(\omega) : \omega \in \Omega\}$, *there exists a probability measure* μ *concentrated on* $\{S_T(\omega) : \omega \in \Omega\}$ *such that* $\mu(\{a\}) > 0$ *and*

$$\mathrm{pr}^i \mu = (1+r)(\varphi^i)'', \quad i = 1, \dots, d$$

($\mathrm{pr}^i \mu$ *denotes the projection of* μ *on the* i*-th coordinate axis of* $\mathbb{R}^d_+$).

`Proof`. *Step 1.* Let us prove the "only if" implication. Take $a \in \{S_T(\omega) : \omega \in \Omega\}$. By Theorem 5.17, there exists $\mathsf{Q} \in \mathcal{R}$ such that $\mathsf{Q}(S_T = a) > 0$. Consider the measure $\mu = \mathrm{Law}_{\mathsf{Q}}\, S_T$. Then $\mu(\{a\}) > 0$, and it follows from the Key Lemma for the model under consideration (it states that $\mathcal{R} = \mathcal{R}\big(\sum_{i=1}^d (S_T^i - (1+r)\varphi^i(0))\big) = \mathcal{M}$) that

$$\int_{\mathbb{R}_+} (x-K)^+ \,\mathrm{pr}^i \mu(dx) = (1+r)\varphi^i(K), \quad i = 1, \dots, d, \ K \in \mathbb{R}_+.$$

By Lemma 3.23 (i),

$$\mathrm{pr}^i \mu = (1+r)(\varphi^i)'', \quad i = 1, \dots, d.$$

Consequently, $(\varphi^i)''(x) > 0$ for any $x \in \{S_T^i(\omega) : \omega \in \Omega\}$, $i = 1, \dots, d$. This implies that the set $\{S_T(\omega) : \omega \in \Omega\}$ is countable. To complete the proof, it remains to note that the measure μ constructed above is concentrated on $\{S_T(\omega) : \omega \in \Omega\}$.

Step 2. Let us prove the "if" implication. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. There exist $\omega_1, \omega_2, \dots \in \Omega$ such that

$$\{S_T(\omega_i) : i = 0, 1, \dots\} = \{S_T(\omega) : \omega \in \Omega\}$$

and

$$S_T(\omega_i) \neq S_T(\omega_j), \quad i, j = 0, 1, \dots, \ i \neq j.$$

Let μ be a probability measure concentrated on $\{S_T(\omega) : \omega \in \Omega\}$ such that $\mu(\{S_T(\omega_0)\}) > 0$ and

$$\mathrm{pr}^i \mu = (1+r)(\varphi^i)'', \quad i = 1, \dots, d.$$

It follows from the Key Lemma for the model under consideration that the measure $\mathsf{Q} = \sum_{k=0} \mu(\{S_T(\omega_k)\})\delta_{\omega_k}$ (the upper limit on the sum may be finite or infinite) is a risk-neutral measure. Moreover, $\mathsf{Q}(D) > 0$. An application of Theorem 5.17 completes the proof. □

Corollary 5.41. *Let* $d = 1$ *and* $\mathbb{K} = \mathbb{R}_+$. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition if and only if the set* $\{S_T(\omega) : \omega \in \Omega\}$ *is countable,* φ'' *is concentrated on this set, and* $\varphi''(\{x\}) > 0$ *for any* $x \in \{S_T(\omega) : \omega \in \Omega\}$.

Proposition 5.42. *Suppose that* $\mathbb{K}^i$ *is finite,* $0 \in \mathbb{K}^i$, $i = 1, \dots, d$, *and*

$$\{S_T(\omega) : \omega \in \Omega\} = \mathbb{R}^d_{++}. \tag{5.25}$$

Then the model $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition if and only if*

(a) φ^i *is strictly positive on* $\mathbb{K}^i$, $i = 1, \dots, d$;
(b) φ^i *is strictly convex on* $\mathbb{K}^i$, $i = 1, \dots, d$;
(c) φ^i *is strictly decreasing on* $\mathbb{K}^i$, $i = 1, \dots, d$;
(d) $(1+r)\varphi^i(K) > (1+r)\varphi^i(0) - K$, $i = 1, \dots, d$, $K \in \mathbb{K}^i \setminus \{0\}$.

`Proof.` *Step 1.* Let us prove the "only if" implication. We will check only (b) (conditions (a), (c), (d) are verified similarly). If (b) is not satisfied, then there exist $i \in \{1,\dots,d\}$ and $K_1 < K_2 < K_3 \in \mathbb{K}^i$ such that

$$\varphi^i(K_2) \ge \frac{K_3 - K_2}{K_3 - K_1}\,\varphi^i(K_1) + \frac{K_2 - K_1}{K_3 - K_1}\,\varphi^i(K_3). \tag{5.26}$$

Set

$$\begin{aligned} X = {} & \frac{K_3 - K_2}{K_3 - K_1}\,(S_T^i - K_1)^+ - (1+r)\frac{K_3 - K_2}{K_3 - K_1}\,\varphi^i(K_1) \\ & - (S_T^i - K_2)^+ + (1+r)\varphi^i(K_2) \\ & + \frac{K_2 - K_1}{K_3 - K_1}\,(S_T^i - K_3)^+ - (1+r)\frac{K_2 - K_1}{K_3 - K_1}\,\varphi^i(K_3). \end{aligned}$$

Then $X \in A$ and, in view of (5.26),

$$X \ge \frac{K_3 - K_2}{K_3 - K_1}\,(S_T^i - K_1)^+ - (S_T^i - K_2)^+ + \frac{K_2 - K_1}{K_3 - K_1}\,(S_T^i - K_3)^+ = g(S_T^i).$$

The function g is positive and $g(K_2) > 0$. Combining this with (5.25), we arrive at a contradiction. As a result, (b) is satisfied.

Step 2. Let us prove the "if" implication. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. Set $a_0^i = S_T^i(\omega_0)$, $i = 1,\dots,d$. For each $i = 1,\dots,d$, we can find a function $\psi^i : \mathbb{R}_+ \to \mathbb{R}_+$ with the properties: ψ^i is convex on $\mathbb{R}_+$; $(\psi^i)'_+(0) = -1$; $\lim_{K\to\infty} \psi^i(K) = 0$; ψ^i is piecewise linear; $(\psi^i)''(\{a_0^i\}) > 0$;

$$\psi^i(K) = (1+r)\varphi^i(K), \quad K \in \mathbb{K}^i.$$

Consider the measure $\mu = (\psi^1)'' \times \dots \times (\psi^d)''$. Then μ is concentrated on a countable set $\{a_0, a_1, \dots\}$, which belongs to $\mathbb{R}^d_{++}$ in view of the condition $(\psi^i)'_+(0) = -1$, $i = 1,\dots,d$. For $k = 1,2,\dots$, find $\omega_k \in \Omega$ such that $S_T(\omega_k) = a_k$. Consider the measure $\mathsf{Q} = \sum_{k=0} \mu(\{S_T(\omega_k)\})\delta_{\omega_k}$. By Lemma 3.23 (ii),

$$\begin{aligned} \mathsf{E_Q}(S_T^i - K)^+ &= \int_{\mathbb{R}_+} (x - K)^+ (\psi^i)''(dx) \\ &= \psi^i(K) = (1+r)\varphi^i(K), \quad i = 1,\dots,d,\ K \in \mathbb{K}^i. \end{aligned}$$

It follows from the Key Lemma for the model under consideration that $\mathsf{Q} \in \mathcal{R}$. Moreover, $\mathsf{Q}(D) > 0$. An application of Theorem 5.17 completes the proof. □

Let us now consider the model with friction, i.e. the model of Section 4.3 with the changes described in Section 5.6. The statement below shows that in most natural cases the NGA condition is satisfied regardless of the structure of $\mathbb{K}^i$.

Proposition 5.43. *Suppose that condition* (5.25) *is satisfied and* $\alpha^i \in (0,1)$, $i = 1,\dots,d$. *If*

(a) $\varphi^{ai} \ge \varphi^{bi}$ *on* $\mathbb{K}^i$, $i = 1,\dots,d$;
(b) φ^{ai} *and* φ^{bi} *are strictly positive on* $\mathbb{K}^i$, $i = 1,\dots,d$;
(c) φ^{ai} *and* φ^{bi} *are strictly convex on* $\mathbb{K}^i$, $i = 1,\dots,d$;

(d) φ^{ai} *and* φ^{bi} *are strictly decreasing on* $\mathbb{K}^i$, $i = 1, \dots, d$;
(e) $(1+r)\varphi^{ai}(K) > (1+r)\varphi^{bi}(0) - K$, $i = 1, \dots, d$, $K \in \mathbb{K}^i \setminus \{0\}$;
(f) $\lim_{K \in \mathbb{K}^i, K \to \infty} \varphi^{ai}(K) = 0$ *if* $\mathbb{K}^i$ *is unbounded from above,* $i = 1, \dots, d$,

then the model $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

`Proof`. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in \Omega$. Set $a_0^i = S_T^i(\omega_0)$, $i = 1, \dots, d$. For each $i = 1, \dots, d$, we can find a function $\varphi^i : \mathbb{R}_+ \to \mathbb{R}_+$ with the properties: φ^i is convex on $\mathbb{R}_+$; $(\varphi^i)'_+(0) = -1$; $\lim_{K \to \infty} \varphi^i(K) = 0$; φ^i is piecewise linear; $(\varphi^i)''(\{a_0^i\}) > 0$;

$$\varphi^i((1-\alpha^i)^{-1}K) \le \frac{1+r}{1-\alpha^i}\,\varphi^{ai}(K), \quad K \in \mathbb{K}^i;$$
$$\varphi^i(K) \ge (1+r)\varphi^{bi}(K), \quad K \in \mathbb{K}^i.$$

Consider the measure $\mu = (\varphi^1)'' \times \dots \times (\varphi^d)''$. The proof is completed in the same way as in Proposition 5.42 (Step 2). □

`Remark`. Propositions 5.42 and 5.43 remain true if we replace condition (5.25) by the condition

$$\{S_T(\omega) : \omega \in \Omega\} = \mathbb{R}_+^d.$$

The proof is the same.

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Index of Notation

$\mathcal{B}(E)$	the Borel σ-field on the space E
C°	the interior or the relative interior of the set C
$\operatorname{conv} C$	the convex hull of the set C
$\overline{\operatorname{conv}}\, C$	the closed convex hull of the set C
$\mathsf{E}_{\mathsf{P}} X$	the P-expectation of the P-integrable or bounded below function X
$\mathsf{E}_{\mathsf{P}}(X \mid \mathcal{G})$	the conditional P-expectation with respect to the σ-field $\mathcal{G}$ of the P-integrable or bounded below function X
$\operatorname{essinf}_{\omega\in\Omega} Z(\omega)$	$\sup\{x \in \mathbb{R} : Z \geq x\ \mathsf{P}\text{-a.s.}\}$, the essential infimum of the function Z
FTAP	fundamental theorem of asset pricing in the probability setting or the possibility setting, 15, 18, 30, 65, 68
$\mathcal{F}_t^X$	$\sigma(X_s; s \leq t)$, the natural filtration of the process X
$I(F)$	the set of fair prices of the contingent claim F in the probability setting or the possibility setting, 15, 19, 66, 70
$I(F_\lambda; \lambda \in \Lambda)$	the set of fair prices of the controlled contingent claim $(F_\lambda)_{\lambda\in\Lambda}$ in the probability setting or the possibility setting, 23, 74
$J(F)$	the set of fair bid-ask prices of the contingent claim F in the probability setting or the possibility setting, 20, 71
$J(F_\lambda; \lambda \in \Lambda)$	the set of fair bid-ask prices of the controlled contingent claim $(F_\lambda)_{\lambda\in\Lambda}$ in the probability setting or the possibility setting, 24, 74
L^0	the space of classes of equivalence under the indistinguishability relation of real-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ in the probability setting or the space of real-valued $\mathcal{F}$-measurable functions in the possibility setting
L^0_+	the set of $\mathbb{R}_+$-valued elements of L^0
$L^1(\mathsf{P})$	the space of P-integrable elements of L^0
L^∞	the space of bounded elements of L^0
$\operatorname{Law}_{\mathsf{P}} X$	the P-distribution of the measurable function X

$\gamma(Z)$	$1 - \text{essinf}_{\omega\in\Omega} Z(\omega)$ in the probability setting or $1 - \inf_{\omega\in\Omega} Z(\omega)$ in the possibility setting, 17, 67
δ_a	the delta-measure concentrated at the point a
$\sigma(E, E')$	the locally convex topology on the space E induced by the duality $\langle\cdot,\cdot\rangle$ between E and E', i.e. the topology generated by the collection of seminorms $\{\lvert\langle\cdot, x'\rangle\rvert; x' \in E'\}$
Φ	the distribution function of the standard normal distribution
$\varphi'_+(x)$	the right-hand derivative of the function φ at the point x
φ''	the second derivative of the convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ taken in the sense of distributions (i.e. $\varphi''((a,b]) = \varphi'_+(b) - \varphi'_+(a)$) with the convention: $\varphi''(\{0\}) = \varphi'_+(0) + 1$
$\sim$	the equivalence relation between probability measures
$\ll$	the absolute continuity relation between probability measures
$\approx$	the approximate equality between sets, 20
$\overset{\text{Law}}{=}$	the equality in law between random variables
$\langle\cdot,\cdot\rangle$	the scalar product in $\mathbb{R}^d$
$\lVert\cdot\rVert$	the Euclidean norm in $\mathbb{R}^d$

Index of Terms