ON STOCHASTIC INTEGRALS UP TO INFINITY AND PREDICTABLE CRITERIA FOR INTEGRABILITY

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Abstract. The first goal of this paper is to give an adequate definition of the stochastic integral

$$\int_0^\infty H_s dX_s, \qquad (*)$$

where $H = (H_t)_{t\geq 0}$ is a predictable process and $X = (X_t)_{t\geq 0}$ is a semimartingale. We consider two different definitions of $(*)$: as a *stochastic integral up to infinity* and as an *improper stochastic integral.*

The second goal of the paper is to give the necessary and sufficient conditions for the existence of the stochastic integral

$$\int_0^t H_s dX_s, \quad t \geq 0$$

and for the existence of the stochastic integral up to infinity $(*)$. These conditions are expressed in predictable terms, i.e. in terms of the predictable characteristics of X.

Moreover, we define the notion of a *semimartingale up to infinity* (*martingale up to infinity*, etc.) and show its connection with the existence of the stochastic integral up to infinity. We also introduce the notion of γ*-localization.*

Key words and phrases. Characteristics of a semimartingale, Fundamental Theorems of Asset Pricing, γ-localization, improper stochastic integrals, Lévy processes, semimartingales up to infinity, stochastic integrals, stochastic integrals up to infinity.

1 Introduction

In the classical analysis there are two approaches to defining the integral $\int_0^\infty h(s)ds$, where h is a Borel function. In the first approach, the improper integral $\int_0^\infty h(s)ds$ is defined as

$$\int_0^\infty h(s)ds := \lim_{t\to\infty}\int_0^t h(s)ds,$$

where $\int_0^t h(s)ds$ is the "usual" Lebesgue integral over $[0,t]$. In the second approach, the integral up to infinity $\int_0^\infty h(s)ds$ is defined as the Lebesgue integral over $[0,\infty)$. Obviously, the classes

$$L = \Big\{h : \forall t \geq 0,\ \int_0^t |h(s)|ds < \infty\Big\}, \tag{1.1}$$

$$L_{\text{imp}} = \Big\{h \in L : \exists \lim_{t\to\infty}\int_0^t h(s)ds\Big\}, \tag{1.2}$$

$$L_{[0,\infty)} = \Big\{h : \int_0^\infty |h(s)|ds < \infty\Big\} \tag{1.3}$$

satisfy the following strict inclusions: $L_{[0,\infty)} \subset L_{\text{imp}} \subset L$.

This paper has two main goals. The first goal is to give the corresponding definitions of the *improper stochastic integral* $\int_0^{\infty} H_s dX_s$ and of the *stochastic integral up to infinity* $\int_0^\infty H_s dX_s$, where $H = H_t(\omega)$ is a predictable process and $X = X_t(\omega)$, $t \geq 0$, $\omega \in \Omega$ is a semimartingale. The second goal is to derive a criterion for the existence of the stochastic integral up to infinity $\int_0^\infty H_s dX_s$ given in the "predictable" terms (see Theorem 4.5). For more information on "predictability", see the monograph [10] by J. Jacod and A.N. Shiryaev. The second edition of this monograph contains a predictable criterion for the existence of stochastic integrals $\int_0^t H_s dX_s$, $t \geq 0$ (see [10; Ch. III, Theorem 6.30]). In this paper, we also derive a predictable criterion for the existence of these integrals (see Theorem 3.2). Our method differs slightly from the one in [10]. As a result, we get a simpler criterion.

The notion of a stochastic integral up to infinity is closely connected with the notion of a *semimartingale up to infinity*. These processes as well as *martingales up to infinity*, etc. are considered in Section 2. C. Stricker [20] also considered "semimartingales jusqu'a l'infini". He used another definition, but it is equivalent to our definition. The notions of a *process with finite variation up to infinity* and *local martingale up to infinity* introduced in Section 2 are closely connected with the notion of γ*-localization* that is also introduced in Section 2.

In Section 3, we recall the definition of a stochastic integral (that is sometimes called the *vector stochastic integral*) and give the predictable criterion for integrability.

Section 4 contains several equivalent definitions of the stochastic integral up to infinity, the definition of the improper stochastic integral as well as the predictable criterion for integrability up to infinity. This criterion is then applied to stable Lévy processes.

In Section 5, we show how the stochastic integrals up to infinity can be used in the mathematical finance (to be more precise, in the Fundamental Theorems of Asset Pricing).

Throughout the paper, a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P}\big)$ is supposed to be fixed. The filtration $(\mathcal{F}_t)$ is assumed to be right-continuous.

2 Semimartingales up to Infinity

1. Notations and definitions. In this section, we consider only one-dimensional processes. The extension to the multidimensional case is straightforward.

We will use the notations $\mathcal{A}_{\mathrm{loc}}$, $\mathcal{V}$, $\mathcal{M}$, $\mathcal{M}_{\mathrm{loc}}$, $\mathcal{S}_p$, and $\mathcal{S}$ for the classes of processes with locally integrable variation, processes with finite variation, martingales, local martingales, special semimartingales, and semimartingales, respectively.

Definition 2.1. We will call a process $Z = (Z_t)_{t\ge 0}$ a *process with locally integrable variation up to infinity* (resp: *process with finite variation up to infinity*, *martingale up to infinity*, *local martingale up to infinity*, *special semimartingale up to infinity*, *semimartingale up to infinity*) if there exists a process $\overline{Z} = (\overline{Z}_t)_{t\ge 0}$ such that

$$\overline{Z}_t = Z_{\frac{t}{1-t}}, \quad t < 1 \tag{2.1}$$

and $\overline{Z}$ is a process with locally integrable variation (resp: process with finite variation, martingale, local martingale, special semimartingale, semimartingale) with respect to the filtration

$$\overline{\mathcal{F}}_t = \begin{cases} \mathcal{F}_{\frac{t}{1-t}}, & t < 1, \\ \mathcal{F}, & t \ge 1. \end{cases} \tag{2.2}$$

We will use the notations $\mathcal{A}_{\mathrm{loc},[0,\infty)}$, $\mathcal{V}_{[0,\infty)}$, $\mathcal{M}_{[0,\infty)}$, $\mathcal{M}_{\mathrm{loc},[0,\infty)}$, $\mathcal{S}_{p,[0,\infty)}$, $\mathcal{S}_{[0,\infty)}$ for these classes of processes.

Note that $\mathcal{A}_{\mathrm{loc},[0,\infty)} \subset \mathcal{A}_{\mathrm{loc}}$, $\mathcal{V}_{[0,\infty)} \subset \mathcal{V}$, $\mathcal{M}_{[0,\infty)} \subset \mathcal{M}$, $\mathcal{M}_{\mathrm{loc},[0,\infty)} \subset \mathcal{M}_{\mathrm{loc}}$, $\mathcal{S}_{p,[0,\infty)} \subset \mathcal{S}_p$, $\mathcal{S}_{[0,\infty)} \subset \mathcal{S}$, and all the inclusions are strict.

2. Basic properties. In the stochastic analysis there exist two types of the "localization" procedure: *localization* (see [10; Ch. I, §1d]) and *σ-localization* (see [10; Ch. III, §6e]). Let us introduce one more type.

Definition 2.2. Let $\mathcal{C}$ be a class of random processes. The *γ-localized* class $\mathcal{C}_\gamma$ consists of the processes X, for which there exists an increasing sequence of stopping times (τ_n) such that $\{\tau_n = \infty\} \uparrow \Omega$ a.s. and, for each n, the stopped process $X_t^{\tau_n} := X_{t\wedge\tau_n}$ belongs to $\mathcal{C}$.

Lemma 2.3. *We have $\mathcal{A}_{\mathrm{loc},[0,\infty)} = \mathcal{A}_\gamma$, where $\mathcal{A}$ is the class of processes with integrable variation.*

The proof is straightforward.

Lemma 2.4. *The class $\mathcal{M}_{[0,\infty)}$ coincides with the class of uniformly integrable martingales.*

`Proof`. This statement follows from the fact that the class of uniformly integrable martingales coincides with the class of the Lévy martingales, i.e. processes Z of the form $Z_t = \mathsf{E}(Z_\infty \mid \mathcal{F}_t)$, $t \ge 0$. □

Lemma 2.5. *The following conditions are equivalent:*
(i) $Z \in \mathcal{M}_{\mathrm{loc},[0,\infty)}$;
(ii) $Z \in (\mathcal{M}_{[0,\infty)})_\gamma$;
(iii) $Z \in \mathcal{M}_{\mathrm{loc}}$ *and* $[Z]^{1/2} \in \mathcal{A}\gamma$.

Proof. **(i)⇒(iii)** This implication follows from the fact that, for a local martingale M, $[M]^{1/2} \in \mathcal{A}_{\mathrm{loc}}$ (see [10; Ch. I, Corollary 4.55]).

(iii)⇒(ii) This implication follows from the Davis inequality (see [16; Ch. I, §9, Theorem 6]).

(ii)⇒(i) This implication is obvious. □

The following statement characterizes the semimartingales as the "L^0-integrators". Recall that a collection of random variables $(\xi_\lambda)_{\lambda\in\Lambda}$ is *bounded in probability* if for any $\varepsilon > 0$, there exists $M > 0$ such that $\mathsf{P}(|\xi_\lambda| > M) < \varepsilon$ for any $\lambda \in \Lambda$. Recall that a stopping time is called *simple* if it takes only a finite number of values, all of which are finite.

Proposition 2.6. *Let Z be a càdlàg $(\mathcal{F}_t)$-adapted process. Then $Z \in \mathcal{S}$ if and only if for any $t \geq 0$, the collection*

$$\left\{\int_0^t H_s dZ_s : H \text{ has the form } \sum_{i=1}^n h_i I_{]\!]S_i,T_i]\!]}, \text{ where } S_1 \leq T_1 \leq \dots \leq S_n \leq T_n \right.$$
$$\left. \text{are simple } (\mathcal{F}_t)\text{-stopping times and } h_i \in [-1,1]\right\}$$

is bounded in probability. (Note that $\int_0^t H_s dZ_s$ here is actually a finite sum.)

For the proof, see [2; Theorem 7.6].

The next statement characterizes the semimartingales up to infinity as the "L^0-integrators up to infinity". Recall that the space $\mathcal{H}^p$ consists of semimartingales Z, for which there exists a decomposition $Z = A + M$ with $A \in \mathcal{V}$, $M \in \mathcal{M}_{\mathrm{loc}}$, and $\mathsf{E}(\operatorname{Var} A)^p_\infty + \mathsf{E}[M]^{p/2}_\infty < \infty$.

Lemma 2.7. *The following conditions are equivalent:*

(i) $Z \in \mathcal{S}_{[0,\infty)}$;

(ii) *there exists a decomposition $Z = A + M$ with $A \in \mathcal{V}_{[0,\infty)}$, $M \in \mathcal{M}_{\mathrm{loc},\,[0,\infty)}$;*

(iii) *there exists an increasing sequence of stopping times (τ_n) such that $\{\tau_n = \infty\} \uparrow \Omega$ a.s. and, for each n, the process $Z_t^{\tau_n -} := Z_t I(t < \tau_n) + Z_{\tau_n -} I(t \geq \tau_n)$ belongs to $\mathcal{H}^1$;*

(iv) *Z is a càdlàg $(\mathcal{F}_t)$-adapted process, and the collection*

$$\left\{\int_0^\infty H_s dZ_s : H \text{ has the form } \sum_{i=1}^n h_i I_{]\!]S_i,T_i]\!]}, \text{ where } S_1 \leq T_1 \leq \dots \leq S_n \leq T_n \right.$$
$$\left. \text{are simple } (\mathcal{F}_t)\text{-stopping times and } h_i \in [-1,1]\right\} \tag{2.3}$$

is bounded in probability. (Note that $\int_0^\infty H_s dZ_s$ here is actually a finite sum.)

Proof. **(i)⇒(ii)** This implication is obvious.

(ii)⇒(iii) It is sufficient to consider the stopping times $\tau_n = \inf\{t \geq 0 : \operatorname{Var} A_t \geq n \text{ or } [M]_t \geq n\}$, where $Z = A + M$ is a decomposition of Z with $A \in \mathcal{V}_{[0,\infty)}$, $M \in \mathcal{M}_{\mathrm{loc},\,[0,\infty)}$.

(iii)$\Rightarrow$(iv) Fix $\varepsilon > 0$. There exists n such that $\mathsf{P}(\tau_n = \infty) > 1 - \varepsilon$. Let $Z^{\tau_n -} = A + M$ be a semimartingale decomposition of $Z^{\tau_n -}$ with $\mathsf{E}(\operatorname{Var} A)_\infty + \mathsf{E}[M]_\infty^{1/2} < \infty$. For any process H of the form described in (2.3), we have

$$\int_0^\infty H_s dZ_s^{\tau_n -} = \int_0^\infty H_s dA_s + \int_0^\infty H_s dM_s.$$

Since $|H| \le 1$, we have $\mathsf{E}|\int_0^\infty H_s dA_s| \le \mathsf{E}(\operatorname{Var} A)_\infty$. It follows from the Davis inequality (see [16; Ch. I, §9, Theorem 6]) that there exists a constant C such that, for any process H of the form described in (2.3), $\mathsf{E}|\int_0^\infty H_s dM_s| \le C$. Combining this with the inequalities

$$\mathsf{P}\biggl(\int_0^\infty H_s dZ_s = \int_0^\infty H_s dZ_s^{\tau_n -}\biggr) \ge \mathsf{P}(\tau_n = \infty) > 1 - \varepsilon,$$

we get (iv).

(iv)$\Rightarrow$(i) For any bounded stopping time S, there exists a sequence of simple stopping times (S_k) such that $S_k \downarrow S$. Hence, the collection

$$\biggl\{\int_0^\infty H_s dZ_s : H \text{ has the form } \sum_{i=1}^n h_i I_{]\!]S_i,T_i]\!]}, \text{ where } S_1 \le T_1 \le \dots \le S_n \le T_n$$
$$\text{are bounded } (\mathcal{F}_t)\text{-stopping times and } h_i \in [-1,1]\biggr\}$$

is bounded in probability.

For $a < b \in \mathbb{Q}$, $n \in \mathbb{N}$, we consider the stopping times

$$\begin{aligned} S_1 &= \inf\{t \ge 0 : Z_t < a\} \wedge n, & T_1 &= \inf\{t \ge S_1 : Z_t > b\} \wedge n, \dots \\ S_n &= \inf\{t \ge T_{n-1} : Z_t < a\} \wedge n, & T_n &= \inf\{t \ge S_n : Z_t > b\} \wedge n. \end{aligned}$$

Take $H^n = \sum_{i=1}^n I_{]\!]S_i,T_i]\!]}$. Then on the set $A := \{Z \text{ upcrosses } [a,b] \text{ infinitely often}\}$ we have

$$\int_0^\infty H_s^n dZ_s \xrightarrow[n\to\infty]{\text{a.s.}} \infty.$$

Hence, $\mathsf{P}(A) = 0$. As a and b have been chosen arbitrarily, we deduce that there exists a limit $Z_\infty := \text{(a.s.)} \lim_{t\to\infty} Z_t$.

Let us set

$$\overline{Z}_t = \begin{cases} Z_{\frac{t}{1-t}}, & t < 1, \\ Z_\infty, & t \ge 1. \end{cases}$$

Using the continuity of $\overline{Z}$ at $t = 1$, one easily verifies that the collection

$$\biggl\{\int_0^1 H_s d\overline{Z}_s : H \text{ has the form } \sum_{i=1}^n h_i I_{]\!]S_i,T_i]\!]}, \text{ where } S_1 \le T_1 \le \dots \le S_n \le T_n$$
$$\text{are simple } (\overline{\mathcal{F}}_t)\text{-stopping times and } h_i \in [-1,1]\biggr\}$$

is bounded in probability (here $(\overline{\mathcal{F}}_t)$ is the filtration given by (2.2)). By Proposition 2.6, $\overline{Z}$ is an $(\overline{\mathcal{F}}_t)$-semimartingale. This means that $Z \in \mathcal{S}_{[0,\infty)}$. $\square$

`Remark`. The description of $\mathcal{S}_{[0,\infty)}$ provided by (iii) is C. Stricker's definition of "semimartingales jusqu'a l'infini" (see [20]). $\square$

3 Stochastic Integrals

1. Notations and definitions. By $\mathcal{A}^d_{\mathrm{loc}}$, $\mathcal{V}^d$, $\mathcal{M}^d$, $\mathcal{M}^d_{\mathrm{loc}}$, $\mathcal{S}^d_p$, $\mathcal{S}^d$ we denote the corresponding spaces of d-dimensional processes.

Let $A \in \mathcal{V}^d$. There exist optional processes a^i and an increasing càdlàg $(\mathcal{F}_t)$-adapted process F such that

$$A^i = A^i_0 + \int_0^{\cdot} a^i_s dF_s. \tag{3.1}$$

Consider the space

$$L_{\mathrm{var}}(A) = \Big\{ H = (H^1, \dots, H^d) : H \text{ is predictable and,} \\ \text{for any } t \geq 0, \int_0^t |H_s \cdot a_s| dF_s < \infty \text{ a.s.} \Big\},$$

where $H_s \cdot a_s := \sum_{i=1}^d H^i_s a^i_s$. Note that $L_{\mathrm{var}}(A)$ does not depend on the choice of a^i and F that satisfy (3.1). For $H \in L_{\mathrm{var}}(A)$, we set

$$\int_0^{\cdot} H_s dA_s := \int_0^{\cdot} H_s \cdot a_s \, dF_s.$$

This is a process with finite variation.

Let $M \in \mathcal{M}^d_{\mathrm{loc}}$. There exist optional processes π^{ij} and an increasing càdlàg $(\mathcal{F}_t)$-adapted process F such that

$$[M^i, M^j] = \int_0^{\cdot} \pi^{ij}_s dF_s. \tag{3.2}$$

Consider the space

$$L^1_{\mathrm{loc}}(M) = \Big\{ H = (H^1, \dots, H^d) : H \text{ is predictable} \\ \text{and } \Big(\int_0^{\cdot} H_s \cdot \pi_s \cdot H_s \, dF_s \Big)^{1/2} \in \mathcal{A}_{\mathrm{loc}} \Big\},$$

where $H_s \cdot \pi_s \cdot H_s := \sum_{i,j=1}^d H^i_s \pi^{ij}_s H^j_s$. Note that $L^1_{\mathrm{loc}}(M)$ does not depend on the choice of π^{ij} and F that satisfy (3.2). For $H \in L^1_{\mathrm{loc}}(M)$, one can define the stochastic integral $\int_0^{\cdot} H_s dM_s$ by the approximation procedure (see [19; Section 3]). This process is a local martingale.

Definition 3.1. Let $X \in \mathcal{S}^d$. A process H is X-*integrable* if there exists a decomposition $X = A + M$ with $A \in \mathcal{V}^d$, $M \in \mathcal{M}^d_{\mathrm{loc}}$ such that $H \in L_{\mathrm{var}}(A) \cap L^1_{\mathrm{loc}}(M)$. In this case

$$\int_0^{\cdot} H_s dX_s := \int_0^{\cdot} H_s dA_s + \int_0^{\cdot} H_s dM_s.$$

The space of X-integrable processes is denoted by $L(X)$.

For the proof of the correctness of this definition and for the basic properties of stochastic integrals, see [19].

2. Predictable criterion for the integrability. Let $X \in \mathcal{S}^d$ and (B, C, ν) be the characteristics of X with respect to the truncation function $xI(|x| \le 1)$ (for the definition, see [10; Ch. II, §2a]). There exist predictable processes b^i, c^{ij}, a transition kernel K from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ (here $\mathcal{P}$ denotes the predictable σ-field) to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and an increasing predictable càdlàg process F such that

$$B^i = \int_0^\cdot b_s^i dF_s, \qquad C^{ij} = \int_0^\cdot c_s^{ij} dF_s, \qquad \nu(\omega, dt, dx) = K(\omega, t, dx) dF_t(\omega) \tag{3.3}$$

(see [10; Ch. II, Proposition 2.9]).

Theorem 3.2. *Let H be a d-dimensional predictable process. Set*

$$\begin{aligned}\varphi_t(H) = \bigg| H_t \cdot b_t + \int_{\mathbb{R}} H_t \cdot x \, \big(I(|x| > 1, |H_t \cdot x| \le 1) - I(|x| \le 1, |H_t \cdot x| > 1) \big) K_t(dx) \bigg| \\ + H_t \cdot c_t \cdot H_t + \int_{\mathbb{R}} 1 \wedge (H_t \cdot x)^2 K_t(dx), \quad t \ge 0. \end{aligned} \tag{3.4}$$

Then $H \in L(X)$ if and only if

$$\forall t \ge 0, \quad \int_0^t \varphi_s(H) dF_s < \infty \quad \text{a.s.} \tag{3.5}$$

The following two statements will be used in the proof.

Proposition 3.3. *Let $X \in \mathcal{S}_p^d$ and $X = X_0 + A + M$ be the canonical decomposition of X. Let $H \in L(X)$. Then $\int_0^\cdot H_s dX_s \in \mathcal{S}_p$ if and only if $H \in L_{\mathrm{var}}(A) \cap L^1_{\mathrm{loc}}(M)$. In this case*

$$\int_0^\cdot H_s dX_s = \int_0^\cdot H_s dA_s + \int_0^\cdot H_s dM_s$$

is the canonical decomposition of $\int_0^\cdot H_s dX_s$.

For the proof, see [9; Proposition 2].

Lemma 3.4. *Let μ be the jump measure of X and $W = W(\omega, t, x)$ be a nonnegative bounded $\mathcal{P} \times \mathcal{B}(\mathbb{R})$-measurable function. Then $(W * \mu)_\infty < \infty$ a.s. if and only if $(W * \nu)_\infty < \infty$ a.s.*

This statement is a direct consequence of the definition of the compensator (see [10; Ch. II, §1a]).

`Remark`. It follows from Lemma 3.4 that

$$\forall t \ge 0, \quad \int_0^t \int_{\mathbb{R}} |H_s \cdot x| I(|x| > 1, |H_s \cdot x| \le 1) K_s(dx) dF_s < \infty \quad \text{a.s.} \tag{3.6}$$

Hence, the process $\varphi(H)$ in (3.5) can be replaced by a simpler process

$$\begin{aligned}\psi_t(H) = \bigg| H_t \cdot b_t - \int_{\mathbb{R}} H_t \cdot x \, I(|x| \le 1, |H_t \cdot x| > 1) K_t(dx) \bigg| \\ + H_t \cdot c_t \cdot H_t + \int_{\mathbb{R}} 1 \wedge (H_t \cdot x)^2 K_t(dx), \quad t \ge 0.\end{aligned}$$

We formulate Theorem 3.2 with the process $\varphi(H)$ and not with $\psi(H)$ in order to achieve the symmetry with the predictable criterion for the integrability up to infinity (Theorem 4.5), where one can use only $\varphi(H)$. □

`Proof of Theorem 3.2.` *The "only if" part.* Let $Y = \int_0^{\cdot} H_s dX_s$. Consider the set $D = \{(\omega, t) : |\Delta X_t(\omega)| > 1 \text{ or } |H_t(\omega) \cdot \Delta X_t(\omega)| > 1\}$. Then D is a.s. discrete, and therefore, the processes

$$\hat{X}^i = \int_0^{\cdot} I_D dX_s^i, \qquad \widetilde{X}^i = X^i - \hat{X}^i,$$
$$\hat{Y} = \int_0^{\cdot} I_D dY_s, \qquad \widetilde{Y} = Y - \hat{Y}$$

are well defined. Obviously, $H \in L_{\mathrm{var}}(\hat{X}) \subseteq L(\hat{X})$. It follows from the equality $\Delta Y = H \cdot \Delta X$ that $\int_0^{\cdot} H_s d\hat{X}_s = \hat{Y}$. By linearity, $H \in L(\widetilde{X})$ and $\int_0^{\cdot} H_s d\widetilde{X}_s = \widetilde{Y}$.

Let μ denote the jump measure of X and X^c denote the continuous martingale part of X. We have

$$X = X_0 + xI(|x| > 1) * \mu + B + xI(|x| \le 1) * (\mu - \nu) + X^c$$

(see [10; Ch. II, Theorem 2.34]). Then

$$\begin{aligned} \widetilde{X} &= X_0 - xI(|x| \le 1, |H \cdot x| > 1) * \mu + B + xI(|x| \le 1) * (\mu - \nu) + X^c \\ &= X_0 + \widetilde{A} + \widetilde{M}, \end{aligned} \tag{3.7}$$

where

$$\widetilde{A} = B - xI(|x| \le 1, |H \cdot x| > 1) * \nu, \tag{3.8}$$
$$\widetilde{M} = xI(|x| \le 1, |H \cdot x| \le 1) * (\mu - \nu) + X^c. \tag{3.9}$$

The process $\widetilde{A}$ is predictable, and therefore, the decomposition $\widetilde{X} = X_0 + \widetilde{A} + \widetilde{M}$ is the canonical decomposition of $\widetilde{X}$. By Proposition 3.3, $H \in L_{\mathrm{var}}(\widetilde{A}) \cap L^1_{\mathrm{loc}}(\widetilde{M})$.

The inclusion $H \in L_{\mathrm{var}}(\widetilde{A})$ means that

$$\forall t \ge 0, \quad \int_0^t \Big| H_s \cdot b_s - \int_{\mathbb{R}} H_s \cdot x\, I(|x| \le 1, |H_s \cdot x| > 1) K_s(dx) \Big| dF_s < \infty \quad \text{a.s.} \tag{3.10}$$

Note that the continuous martingale part of $\widetilde{M}$ is X^c. Consequently,

$$[\widetilde{M}^i, \widetilde{M}^j] = \sum_{s \le \cdot} \Delta \widetilde{M}_s^i \Delta \widetilde{M}_s^j + \int_0^{\cdot} c_s^{ij} dF_s \tag{3.11}$$

(see [10; Ch. I, Theorem 4.52]). Now, the inclusion $H \in L^1_{\mathrm{loc}}(\widetilde{M})$ implies that

$$\forall t \ge 0, \quad \int_0^t H_s \cdot c_s \cdot H_s \, dF_s < \infty \quad \text{a.s.} \tag{3.12}$$

We have

$$\forall t \ge 0, \quad \sum_{s \le t} (H_s \cdot \Delta X_s)^2 = \sum_{s \le t} \Delta Y_s^2 < \infty \quad \text{a.s.},$$

and using Lemma 3.4, we obtain

$$\forall t \geq 0, \quad \int_0^t \int_{\mathbb{R}} 1 \wedge (H_s \cdot x)^2 K_s(dx) dF_s < \infty \quad \text{a.s.} \tag{3.13}$$

Inequalities (3.6), (3.10), (3.12), and (3.13) taken together yield (3.5).

The "if" part. Combining condition (3.5) with Lemma 3.4, we get

$$\forall t \geq 0, \quad \sum_{s \leq t} (H_s \cdot \Delta X_s)^2 < \infty \quad \text{a.s.} \tag{3.14}$$

Hence, the set D introduced above is a.s. discrete. The processes $\hat{X}$ and $\widetilde{X}$ are well defined, and equalities (3.7)–(3.9) hold true.

Condition (3.5), combined with (3.6), implies (3.10), which means that $H \in L_{\text{var}}(\widetilde{A})$.

It follows from (3.14) that

$$\forall t \geq 0, \quad \sum_{s \leq t} (H_s \cdot \Delta \widetilde{X}_s)^2 < \infty \quad \text{a.s.}$$

The inclusion $H \in L_{\text{var}}(\widetilde{A})$ implies that

$$\forall t \geq 0, \quad \sum_{s \leq t} |H_s \cdot \Delta \widetilde{A}_s| < \infty \quad \text{a.s.}$$

Taking into account the equality $\Delta \widetilde{M} = \Delta \widetilde{X} - \Delta \widetilde{A}$, we get

$$\forall t \geq 0, \quad \sum_{s \leq t} (H_s \cdot \Delta \widetilde{M}_s)^2 < \infty \quad \text{a.s.}$$

Moreover, $H \cdot \Delta \widetilde{A} = {}^p(H \cdot \Delta \widetilde{X})$ (see [10; Ch. I, §2d]), which implies that $|H \cdot \Delta \widetilde{A}| \leq 1$, and hence, $|H \cdot \Delta \widetilde{M}| \leq 2$. Consequently,

$$\left(\sum_{s \leq \cdot} (H_s \cdot \Delta \widetilde{M}_s)^2 \right)^{1/2} \in \mathcal{A}_{\text{loc}}.$$

This, combined with (3.11) and (3.12) (that is a consequence of (3.5)), yields the inclusion $H \in L^1_{\text{loc}}(\widetilde{M})$.

As a result, $H \in L(\widetilde{X})$. Since obviously $H \in L_{\text{var}}(\hat{X}) \subseteq L(\hat{X})$, we get $H \in L(X)$. □

Corollary 3.5. *Let X be a one-dimensional continuous semimartingale with the canonical decomposition $X = X_0 + A + M$. Then a predictable process H belongs to $L(X)$ if and only if*

$$\forall t \geq 0, \quad \int_0^t |H_s| d(\operatorname{Var} A)_s + \int_0^t H_s^2 d\langle M \rangle_s < \infty \quad \text{a.s.}$$

3. The application to Lévy processes. Let X be a one-dimensional $(\mathcal{F}_t)$-Lévy process, i.e. X is an $(\mathcal{F}_t)$-adapted Lévy process and, for any $s \le t$, the increment $X_t - X_s$ is independent of $\mathcal{F}_s$. The notation $X \sim (b,c,\nu)_h$ means that

$$\mathsf{E}e^{i\lambda X_t} = \exp\Big\{t\Big[i\lambda b - \frac{\lambda^2}{2}c + \int_{\mathbb{R}} \big(e^{i\lambda x} - 1 - i\lambda h(x)\big)\nu(dx)\Big]\Big\}.$$

For more information on Lévy processes, see [18].

The following corollary of Theorem 3.2 completes the results of O. Kallenberg [12], [13], J. Kallsen and A.N. Shiryaev [15], J. Rosinski and W. Woyczynski [17].

Corollary 3.6. *Let X be an α-stable $(\mathcal{F}_t)$-Lévy process with the Lévy measure*

$$\nu(dx) = \Big(\frac{m_1 I(x<0)}{|x|^{\alpha+1}} + \frac{m_2 I(x>0)}{|x|^{\alpha+1}}\Big)dx.$$

Let H be a predictable process.

(i) *Let $\alpha \in (0,1)$ and $X \sim (b,0,\nu)_0$. Then $H \in L(X)$ if and only if*

$$\forall t \ge 0, \quad |b|\int_0^t |H_s|ds + (m_1+m_2)\int_0^t |H_s|^\alpha ds < \infty \quad a.s.$$

(ii) *Let $\alpha = 1$ and $X \sim (b,0,\nu)_h$, where $h(x) = xI(|x|\le 1)$. Then $H \in L(X)$ if and only if*

$$\forall t \ge 0, \quad (|b| + m_1 + m_2)\int_0^t |H_s|ds + |m_1 - m_2|\int_0^t |H_s|\ln|H_s|ds < \infty \quad a.s.$$

(iii) *Let $\alpha \in (1,2)$ and $X \sim (b,0,\nu)_x$. Then $H \in L(X)$ if and only if*

$$\forall t \ge 0, \quad |b|\int_0^t |H_s|ds + (m_1+m_2)\int_0^t |H_s|^\alpha ds < \infty \quad a.s.$$

(iv) *Let $\alpha = 2$ and $X \sim (b,c,0)$. Then $H \in L(X)$ if and only if*

$$\forall t \ge 0, \quad |b|\int_0^t |H_s|ds + c\int_0^t H_s^2 ds < \infty \quad a.s.$$

`Proof`. The case $\alpha = 2$ is obvious. Let us consider the case $\alpha \in (0,2)$. The semimartingale characteristics (B', C', ν') of X with respect to the truncation function $xI(|x|\le 1)$ are given by

$$B'_t = b't, \qquad C'_t = 0, \qquad \nu'(\omega, dt, dx) = \nu(dx)dt.$$

The value b' is specified below. We have

$$\int_{\mathbb{R}} 1 \wedge (Hx)^2 \nu(dx) = \frac{2(m_1+m_2)}{\alpha(2-\alpha)}|H|^\alpha, \quad H \in \mathbb{R}.$$

In case (i), we have

$$b' = b + \int_{\mathbb{R}} xI(|x|\le 1)\nu(dx)$$

and

$$
\begin{aligned}
&Hb' + \int_{\mathbb{R}} Hx\big(I(|x| > 1, |Hx| \le 1) - I(|x| \le 1, |Hx| > 1)\big)\nu(dx)\\
&= Hb + \operatorname{sgn} H\,\frac{m_2 - m_1}{1-\alpha}\,|H|^{\alpha}, \quad H \in \mathbb{R}.
\end{aligned}
$$

In case (ii), we have $b' = b$ and

$$
\begin{aligned}
&Hb' + \int_{\mathbb{R}} Hx\big(I(|x| > 1, |Hx| \le 1) - I(|x| \le 1, |Hx| > 1)\big)\nu(dx)\\
&= Hb + (m_1 - m_2)H \ln |H|, \quad H \in \mathbb{R}.
\end{aligned}
$$

In case (iii), we have

$$
b' = b - \int_{\mathbb{R}} xI(|x| > 1)\nu(dx)
$$

and

$$
\begin{aligned}
&Hb' + \int_{\mathbb{R}} Hx\big(I(|x| > 1, |Hx| \le 1) - I(|x| \le 1, |Hx| > 1)\big)\nu(dx)\\
&= Hb + \operatorname{sgn} H\,\frac{m_2 - m_1}{1-\alpha}\,|H|^{\alpha}, \quad H \in \mathbb{R}.
\end{aligned}
$$

The result now follows from Theorem 3.2. □

Corollary 3.7. *Let X be a nondegenerate strictly α-stable $(\mathcal{F}_t)$-Lévy process. Then a predictable process H belongs to $L(X)$ if and only if*

$$
\forall t \ge 0, \quad \int_0^t |H_s|^{\alpha} ds < \infty \quad \text{a.s.}
$$

Corollary 3.8. *Let X be an $(\mathcal{F}_t)$-Lévy process, whose diffusion component is not equal to zero. Then a predictable process H belongs to $L(X)$ if and only if*

$$
\forall t \ge 0, \quad \int_0^t H_s^2 ds < \infty \quad \text{a.s.}
$$

`Proof`. This statement follows from Theorem 3.2 and the estimates

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} Hx\big(I(|x| > 1, |Hx| \le 1) - I(|x| \le 1, |Hx| > 1)\big)\nu(dx)\right|\\
&\le \int_{\mathbb{R}} I(|x| > 1)\nu(dx) + H^2 \int_{\mathbb{R}} 1 \wedge x^2 \nu(dx), \quad H \in \mathbb{R},
\end{aligned}
$$

$$
\int_{\mathbb{R}} 1 \wedge (Hx)^2 \nu(dx) \le (1 \vee H^2) \int_{\mathbb{R}} 1 \wedge x^2 \nu(dx), \quad H \in \mathbb{R}.
$$

□

4 Stochastic Integrals up to Infinity and Improper Stochastic Integrals

1. Various definitions. Let $A \in \mathcal{V}^d$ and a^i, F satisfy (3.1). Consider the space

$$L_{\mathrm{var},\,[0,\infty)}(A) = \Big\{ H = (H^1, \dots, H^d) : H \text{ is predictable} \\ \text{and } \int_0^\infty |H_s \cdot a_s| dF_s < \infty \text{ a.s.} \Big\}.$$

For $H \in L_{\mathrm{var},\,[0,\infty)}(A)$, we set

$$\int_0^\infty H_s dA_s := \int_0^\infty H_s \cdot a_s\, dF_s.$$

Let $M \in \mathcal{M}^d_{\mathrm{loc}}$ and π^{ij}, F satisfy (3.2). Consider the space

$$L^1_{\mathrm{loc},\,[0,\infty)}(M) = \Big\{ H = (H^1, \dots, H^d) : H \text{ is predictable} \\ \text{and } \Big(\int_0^\cdot H_s \cdot \pi_s \cdot H_s\, dF_s \Big)^{1/2} \in \mathcal{A}_{\mathrm{loc},\,[0,\infty)} \Big\}.$$

For $H \in L^1_{\mathrm{loc},\,[0,\infty)}(M)$, one can define $\int_0^\infty H_s dM_s$ by the approximation procedure similarly to the definition of $\int_0^t H_s dM_s$.

Definition 4.1. Let $X \in \mathcal{S}^d$. We will say that a process H is *X-integrable up to infinity* if there exists a decomposition $X = A + M$ with $A \in \mathcal{V}^d$, $M \in \mathcal{M}^d_{\mathrm{loc}}$ such that $H \in L_{\mathrm{var},\,[0,\infty)}(A) \cap L^1_{\mathrm{loc},\,[0,\infty)}(M)$. In this case

$$\int_0^\infty H_s dX_s := \int_0^\infty H_s dA_s + \int_0^\infty H_s dM_s.$$

The space of X-integrable up to infinity processes will be denoted by $L_{[0,\infty)}(X)$.

`Remark`. The above definition of $\int_0^\infty H_s dX_s$ is correct, i.e. it does not depend on the choice of the decomposition $X = A + M$. Indeed, it follows from the definition of $\int_0^\infty H_s dA_s$ and $\int_0^\infty H_s dM_s$ that

$$\int_0^\infty H_s dX_s = \text{(a.s.)} \lim_{t \to \infty} \int_0^t H_s dX_s. \tag{4.1}$$

Theorem 4.2. *Let $X \in \mathcal{S}^d$. Then $H \in L_{[0,\infty)}(X)$ if and only if $H \in L(X)$ and $\int_0^\cdot H_s dX_s \in \mathcal{S}_{[0,\infty)}$.*

`Proof`. *The "only if" part.* The inclusion $H \in L_{\mathrm{var},\,[0,\infty)}(A)$ implies that $H \in L_{\mathrm{var}}(A)$ and $\int_0^\cdot H_s dA_s \in \mathcal{V}_{[0,\infty)} \subset \mathcal{S}_{[0,\infty)}$. The inclusion $H \in L^1_{\mathrm{loc},\,[0,\infty)}(M)$ implies that $H \in L^1_{\mathrm{loc}}(M)$, $\int_0^\cdot H_s dM_s \in \mathcal{M}_{\mathrm{loc}}$, and

$$\Big[\int_0^\cdot H_s dM_s \Big]^{1/2} = \Big(\int_0^\cdot H_s \cdot \pi_s \cdot H_s\, dF_s \Big)^{1/2} \in \mathcal{A}_{\mathrm{loc},\,[0,\infty)}$$

(the equality here follows from [19; Lemma 4.18]). In view of Lemma 2.5, $\int_0^\cdot H_s dM_s \in \mathcal{M}_{\mathrm{loc},\,[0,\infty)} \subset \mathcal{S}_{[0,\infty)}$. As a result, $H \in L(X)$ and $\int_0^\cdot H_s dX_s \in \mathcal{S}_{[0,\infty)}$.

The "if" part. Proposition 2.6 and Lemma 2.7 combined together show that one can find deterministic functions $K^1, \dots, K^d$ such that, for each i, $K^i > 0$, $K^i \in L(X^i)$ and $Y^i := \int_0^\cdot (K_s^i)^{-1} dX_s^i \in \mathcal{S}_{[0,\infty)}$. It follows from the associativity property of stochastic integrals (see [19; Theorem 4.7]) that the process $J = (K^1 H^1, \dots, K^d H^d)$ belongs to $L(Y)$ and, for the process $Z = \int_0^\cdot J_s dY_s$, we have $Z = \int_0^\cdot H_s dX_s \in \mathcal{S}_{[0,\infty)}$. Set

$$\overline{Y}_t = \begin{cases} Y_{\frac{t}{1-t}}, & t < 1, \\ Y_\infty, & t \ge 1, \end{cases} \qquad \overline{Z}_t = \begin{cases} Z_{\frac{t}{1-t}}, & t < 1, \\ Z_\infty, & t \ge 1, \end{cases} \qquad \overline{J}_t = \begin{cases} J_{\frac{t}{1-t}}, & t < 1, \\ 0, & t \ge 1. \end{cases}$$

Let us prove that $\overline{J} \in L(\overline{Y})$. It will suffice to verify (see [19; Lemma 4.13]) that, for any sequences $a_n < b_n$ with $a_n \to \infty$ and any sequence $(\overline{G}^n)$ of one-dimensional $(\overline{\mathcal{F}}_t)$-predictable processes with $|\overline{G}^n| \le 1$ (here $(\overline{\mathcal{F}}_t)$ is the filtration given by (2.2)), we have

$$\int_0^{1-1/n} \overline{G}_s^n \overline{J}_s I(a_n < |\overline{J}_s| \le b_n) d\overline{Y}_s \xrightarrow[n\to\infty]{\mathsf{P}} 0. \tag{4.2}$$

We can write

$$\begin{aligned} \int_0^{1-1/n} \overline{G}_s^n \overline{J}_s I(a_n < |\overline{J}_s| \le b_n) d\overline{Y}_s &= \int_0^{n-1} G_s^n J_s I(a_n < |J_s| \le b_n) dY_s \\ = \int_0^{n-1} G_s^n I(a_n < |J_s| \le b_n) dZ_s &= \int_0^{1-1/n} \overline{G}_s^n I(a_n < |\overline{J}_s| \le b_n) d\overline{Z}_s, \end{aligned}$$

where $G_t^n = \overline{G}_{t/1+t}^n$. Using the dominated convergence theorem for stochastic integrals (see [10; Ch. I, Theorem 4.40]), we get (4.2).

Thus, $\overline{J} \in L(\overline{Y})$, which means that there exists a decomposition $\overline{Y} = \overline{B} + \overline{N}$ with $\overline{B} \in \mathcal{V}^d(\overline{\mathcal{F}}_t)$, $\overline{N} \in \mathcal{M}_{\mathrm{loc}}^d(\overline{\mathcal{F}}_t)$ such that $\overline{J} \in L_{\mathrm{var}}(\overline{B}) \cap L_{\mathrm{loc}}^1(\overline{N})$. Then $J \in L_{\mathrm{var},\,[0,\infty)}(B) \cap L^1_{\mathrm{loc},\,[0,\infty)}(M)$, where $B_t = \overline{B}_{t/1+t}$, $N_t = \overline{N}_{t/1+t}$. Consequently, $H \in L_{\mathrm{var},\,[0,\infty)}(A) \cap L^1_{\mathrm{loc},\,[0,\infty)}(M)$, where

$$A^i = \int_0^\cdot K_s^i dB_s^i, \qquad M^i = \int_0^\cdot K_s^i dN_s^i.$$

Since $X = X_0 + A + M$, the proof is completed. □

Let us compare the notion of a stochastic integral up to infinity introduced above with the notion of an improper stochastic integral introduced below.

Definition 4.3. Let $X \in \mathcal{S}^d$. We will say that a process H is *improperly X-integrable* if $H \in L(X)$ and there exists a limit

$$\text{(a.s.)} \lim_{t\to\infty} \int_0^t H_s dX_s.$$

This limit is called the *improper stochastic integral* $\int_0^\infty H_s dX_s$. The space of improperly X-integrable processes will be denoted by $L_{\mathrm{imp}}(X)$.

By the definition, $L_{\mathrm{imp}}(X) \subseteq L(X)$. It follows from Theorem 4.2 and equality (4.1) that $L_{[0,\infty)}(X) \subseteq L_{\mathrm{imp}}(X)$ and the two definitions of $\int_0^\infty H_s dX_s$ coincide for $H \in L_{[0,\infty)}(X)$. The following example shows that these two inclusions are strict.

Example 4.4. *Let $X_t = t$ and $H_t = h(t)$ be a measurable deterministic function. Then*

$$\begin{aligned} H \in L(X) &\iff h \in L, \\ H \in L_{\mathrm{imp}}(X) &\iff h \in L_{\mathrm{imp}}, \\ H \in L_{[0,\infty)}(X) &\iff h \in L_{[0,\infty)}, \end{aligned}$$

where the classes L, L_{imp}, and $L_{[0,\infty)}$ are defined in (1.1)–(1.3).

Proof. The first two statements follow from Theorem 3.2. The third statement is a consequence of Theorem 4.5. □

2. Predictable criterion for the integrability up to infinity. The following statement provides a description of $L_{[0,\infty)}(X)$ that is "dual" to the description of $L(X)$ provided by Theorem 3.2. We use the notation from Subsection 3.2.

Theorem 4.5. *Let H be a d-dimensional predictable process. Then $H \in L_{[0,\infty)}(X)$ if and only if*

$$\int_0^\infty \varphi_s(H) dF_s < \infty \quad \text{a.s.}, \tag{4.3}$$

where $\varphi(H)$ is given by (3.4).

Proposition 4.6. *Let $H \in L(X)$. Then the characteristics $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})$ of $\int_0^\cdot H_s dX_s$ with respect to the truncation function $xI(|x| \le 1)$ are given by*

$$\begin{aligned} \widetilde{B} = \int_0^\cdot \Big(H_s \cdot b_s + \int_{\mathbb{R}} H_s \cdot x \big(I(|x| > 1, |H_s \cdot x| \le 1) \\ - I(|x| \le 1, |H_s \cdot x| > 1) \big) K_s(dx) \Big) dF_s, \end{aligned} \tag{4.4}$$

$$\widetilde{C} = \int_0^\cdot H_s \cdot c_s \cdot H_s \, dF_s, \tag{4.5}$$

$$\widetilde{\nu}(\omega, dt, dx) = \widetilde{K}(\omega, t, dx) dF_t(\omega), \tag{4.6}$$

where $\widetilde{K}(\omega, t, dx)$ is the image of $K(\omega, t, dx)$ under the map $\mathbb{R}^d \ni x \mapsto H_t(\omega) \cdot x \in \mathbb{R}$ and b, c, K, F satisfy (3.3).

For the proof, see [15; Lemma 3].

Proof of Theorem 4.5. *The "only if" part.* The process $Y = \int_0^\cdot H_s dX_s$ is a semimartingale up to infinity. Hence, the process

$$\widetilde{Y} = Y - \sum_{s \le \cdot} \Delta Y_s I(|\Delta Y_s| > 1) \tag{4.7}$$

is also a semimartingale up to infinity. Since $\widetilde{Y}$ has bounded jumps, it belongs to $\mathcal{S}_{p,\,[0,\infty)}$, and therefore, its canonical decomposition $\widetilde{Y} = \widetilde{B} + \widetilde{N}$ satisfies $\widetilde{B} \in \mathcal{V}_{[0,\infty)}$, $\widetilde{N} \in \mathcal{M}_{\mathrm{loc},\,[0,\infty)}$.

The process $\widetilde{B}$ is given by (4.4). The inclusion $\widetilde{B} \in \mathcal{V}_{[0,\infty)}$ means that

$$\begin{aligned}\int_0^\infty \Big| H_s \cdot b_s + \int_{\mathbb{R}} H_s \cdot x \big(I(|x| > 1, |H_s \cdot x| \le 1) \\ - I(|x| \le 1, |H_s \cdot x| > 1) \big) K_s(dx) \Big| dF_s < \infty \quad \text{a.s.}\end{aligned} \tag{4.8}$$

The inclusion $\widetilde{N} \in \mathcal{M}_{\mathrm{loc},\,[0,\infty)}$ implies that $[\widetilde{N}]_\infty < \infty$ a.s. Therefore, $\langle \widetilde{N}^c \rangle < \infty$ a.s. (see [10; Ch. I, Theorem 4.52]). In view of (4.5), this means that

$$\int_0^\infty H_s \cdot c_s \cdot H_s \, dF_s < \infty \quad \text{a.s.} \tag{4.9}$$

We have

$$\sum_{s \ge 0} (H_s \cdot \Delta X_s)^2 = \sum_{s \ge 0} \Delta Y_s^2 < \infty \quad \text{a.s.,}$$

and using Lemma 3.4, we obtain

$$\int_0^\infty \int_{\mathbb{R}} 1 \wedge (H_s \cdot x)^2 K_s(dx) dF_s < \infty \quad \text{a.s.} \tag{4.10}$$

Inequalities (4.8)–(4.10) taken together yield (4.3).

The "if" part. It follows from Theorem 3.2 that $H \in L(X)$. Set $Y = \int_0^\cdot H_s dX_s$ and define $\widetilde{Y}$ by (4.7). The process $\widetilde{B}$ in the canonical decomposition $\widetilde{Y} = \widetilde{B} + \widetilde{N}$ is given by (4.4).

Condition (4.3) implies (4.8), which means that $\widetilde{B} \in \mathcal{V}_{[0,\infty)}$.

Combining condition (4.3) with Lemma 3.4, we get

$$\sum_{s \ge 0} \Delta \widetilde{Y}_s^2 = \sum_{s \ge 0} (H_s \cdot \Delta X_s)^2 I(|H_s \cdot \Delta X_s| \le 1) < \infty \quad \text{a.s.}$$

The inclusion $\widetilde{B} \in \mathcal{V}_{[0,\infty)}$ implies that

$$\sum_{s \ge 0} |\Delta \widetilde{B}_s| < \infty \quad \text{a.s.}$$

Taking into account the equality $\Delta \widetilde{N} = \Delta \widetilde{Y} - \Delta \widetilde{B}$, we get

$$\sum_{s \ge 0} \Delta \widetilde{N}_s^2 < \infty \quad \text{a.s.}$$

Since $|\Delta \widetilde{N}| \le 2$ (see [10; Ch. I, Lemma 4.24]), we have

$$\Big(\sum_{s \le \cdot} \Delta \widetilde{N}_s^2 \Big)^{1/2} \in \mathcal{A}_{\mathrm{loc},\,[0,\infty)}.$$

In view of (4.5),

$$\langle \widetilde{N}^c \rangle_\infty = \int_0^\infty H_s \cdot c_s \cdot H_s \, dF_s < \infty \quad \text{a.s.}$$

Thus, $[\widetilde{N}]^{1/2} \in \mathcal{A}_{\mathrm{loc},\,[0,\infty)}$. By Lemma 2.5, $\widetilde{N} \in \mathcal{M}_{\mathrm{loc},\,[0,\infty)}$.

As a result, $\widetilde{Y} \in \mathcal{S}_{[0,\infty)}$. Moreover, condition (4.3), together with Lemma 3.4, implies that

$$\sum_{s\geq 0} I(|\Delta Y_s| > 1) = \sum_{s\geq 0} I(|H_s \cdot \Delta X_s| > 1) < \infty \quad \text{a.s.}$$

Hence, $Y \in \mathcal{S}_{[0,\infty)}$. By Theorem 4.2, $H \in L_{[0,\infty)}(X)$. $\square$

Corollary 4.7. *Let X be a one-dimensional continuous semimartingale with the canonical decomposition $X = X_0 + A + M$. Then a predictable process H belongs to $L_{[0,\infty)}(X)$ if and only if*

$$\int_0^\infty |H_s| d(\mathrm{Var}\, A)_s + \int_0^\infty H_s^2 d\langle M\rangle_s < \infty \quad \text{a.s.}$$

3. The application to Lévy processes. The following statement is "dual" to Corollary 3.6.

Corollary 4.8. *Let X be an α-stable $(\mathcal{F}_t)$-Lévy process with the Lévy measure*

$$\nu(dx) = \left(\frac{m_1 I(x<0)}{|x|^{\alpha+1}} + \frac{m_2 I(x>0)}{|x|^{\alpha+1}}\right) dx.$$

Let H be a predictable process.

(i) *Let $\alpha \in (0,1)$ and $X \sim (b,0,\nu)_0$. Then $H \in L_{[0,\infty)}(X)$ if and only if*

$$|b| \int_0^\infty |H_s| ds + (m_1 + m_2) \int_0^\infty |H_s|^\alpha ds < \infty \quad a.s.$$

(ii) *Let $\alpha = 1$ and $X \sim (b,0,\nu)_h$, where $h(x) = xI(|x| \leq 1)$. Then $H \in L_{[0,\infty)}(X)$ if and only if*

$$(|b| + m_1 + m_2) \int_0^\infty |H_s| ds + |m_1 - m_2| \int_0^\infty |H_s| \ln |H_s| ds < \infty \quad a.s.$$

(iii) *Let $\alpha \in (1,2)$ and $X \sim (b,0,\nu)_x$. Then $H \in L_{[0,\infty)}(X)$ if and only if*

$$|b| \int_0^\infty |H_s| ds + (m_1 + m_2) \int_0^\infty |H_s|^\alpha ds < \infty \quad a.s.$$

(iv) *Let $\alpha = 2$ and $X \sim (b,c,0)$. Then $H \in L_{[0,\infty)}(X)$ if and only if*

$$|b| \int_0^\infty |H_s| ds + c \int_0^\infty H_s^2 ds < \infty \quad a.s.$$

The proof is similar to the proof of Corollary 3.6.

Corollary 4.9. *Let X be a nondegenerate strictly α-stable $(\mathcal{F}_t)$-Lévy process. Then a predictable process H belongs to $L_{[0,\infty)}(X)$ if and only if*

$$\int_0^\infty |H_s|^\alpha ds < \infty \quad \text{a.s.}$$

5 Application to Mathematical Finance

1. Fundamental Theorems of Asset Pricing. Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P}; (X_t)_{t\geq 0}\big)$ be a *model* of a financial market. Here X is a multidimensional $(\mathcal{F}_t)$-semimartingale. From the financial point of view, X is the discounted price process of several assets. Recall that a *strategy* is a pair (x, H), where $x \in \mathbb{R}$ and $H \in L(X)$. The discounted *capital* of this strategy is $x + \int_0^\cdot H_s dX_s$.

The following notion was introduced by F. Delbaen and W. Schachermayer [4]. We formulate it using the notion of an improper stochastic integral introduced above (see Definition 4.3).

Definition 5.1. A sequence of strategies (x^n, H^n) realizes *free lunch with vanishing risk* if

(i) for each n, $x^n = 0$;

(ii) for each n, there exists $a_n \in \mathbb{R}$ such that $\int_0^\cdot H_s^n dX_s \geq a_n$ a.s.;

(iii) for each n, $H^n \in L_{\text{imp}}(X)$;

(iv) for each n, $\int_0^\infty H_s^n dX_s \geq -\frac{1}{n}$ a.s.;

(v) there exists $\delta > 0$ such that, for each n, $\mathsf{P}\big(\int_0^\infty H_s^n dX_s > \delta\big) > \delta$.

A model satisfies the *no free lunch with vanishing risk* condition if such a sequence of strategies does not exist. Notation: (*NFLVR*) .

Recall that a one-dimensional process X is called a σ*-martingale* if there exists a sequence of predictable sets (D_n) such that $D_n \subseteq D_{n+1}$, $\bigcup_n D_n = \Omega \times \mathbb{R}_+$ and, for any n, the process $\int_0^\cdot I_{D_n} dX_s$ is a uniformly integrable martingale. For more information on σ-martingales, see [3], [6], [7], [10; Ch. III, §6e], [14], [19]. A multidimensional process X is called a σ-martingale if each its component is a σ-martingale. The space of d-dimensional σ-martingales is denoted by $\mathcal{M}_\sigma^d$.

Proposition 5.2 (First Fundamental Theorem of Asset Pricing). *A model satisfies the* (*NFLVR*) *condition if and only if there exists an equivalent* σ*-martingale measure, i.e. a measure* $\mathsf{Q} \sim \mathsf{P}$ *such that* $X \in \mathcal{M}_\sigma^d(\mathcal{F}_t, \mathsf{Q})$.

This theorem was proved by F. Delbaen and W. Schachermayer [5] (compare with Yu.M. Kabanov [11]).

Definition 5.3. A model is *complete* if for any bounded $\mathcal{F}$-measurable function f, there exists a strategy (x, H) such that

(i) there exist constants a, b such that $a \leq \int_0^\cdot H_s dX_s \leq b$ a.s.;

(ii) $H \in L_{\text{imp}}(X)$;

(iii) $f = x + \int_0^\infty H_s dX_s$ a.s.

Proposition 5.4 (Second Fundamental Theorem of Asset Pricing). *Suppose that a model satisfies the* (*NFLVR*) *condition. Then it satisfies the completeness condition if and only if the equivalent* σ*-martingale measure is unique.*

This statement follows from [5; Theorem 5.14]. It can also be derived from [1] or [8; Théorème 11.2]. An explicit proof of the Second Fundamental Theorem of Asset Pricing in this form can be found in [19].

2. Stochastic integrals up to infinity in the Fundamental Theorems of Asset Pricing. If condition (iii) of Definition 5.1 and condition (ii) of Definition 5.3 are replaced by the conditions

(iii)' for each n, $H^n \in L_{[0,\infty)}(X)$,

(ii)' $H \in L_{[0,\infty)}(X)$,

respectively, then new versions of the (*NFLVR*) and of the completeness are obtained. We assert that the First and the Second Fundamental Theorems of Asset Pricing remain valid with these new versions of the (*NFLVR*) and of the completeness.

Theorem 5.5. *A model satisfies the* (*NFLVR*) *condition with the stochastic integrals up to infinity if and only if there exists an equivalent σ-martingale measure.*

Theorem 5.6. *Suppose that a model satisfies the* (*NFLVR*) *condition with the stochastic integrals up to infinity. Then it satisfies the completeness condition with the stochastic integrals up to infinity if and only if the equivalent σ-martingale measure is unique.*

`Proof of Theorems 5.5, 5.6.` It follows from Proposition 2.6 and Lemma 2.7 that there exist deterministic functions $K^1,\dots,K^d$ such that, for each i, $K^i > 0$, $K^i \in L(X^i)$ and $Y^i := \int_0^\cdot (K^i_s)^{-1} dX^i_s \in \mathcal{S}_{[0,\infty)}$. Set

$$\overline{Y}_t = \begin{cases} Y_{\frac{t}{1-t}}, & t<1, \\ Y_\infty, & t \ge 1, \end{cases} \qquad \overline{\mathcal{F}}_t = \begin{cases} \mathcal{F}_{\frac{t}{1-t}}, & t<1, \\ \mathcal{F}, & t \ge 1. \end{cases}$$

Then each of the following conditions

(*NFLVR*) with the stochastic integrals up to infinity;

existence of an equivalent σ-martingale measure;

completeness with the stochastic integrals up to infinity;

uniqueness of an equivalent σ-martingale measure

holds or does not hold for the following models simultaneously

$$\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathsf{P}; (X_t)_{t\ge 0}\big),$$
$$\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathsf{P}; (Y_t)_{t\ge 0}\big),$$
$$\big(\Omega, \mathcal{F}, (\overline{\mathcal{F}}_t)_{t\ge 0}, \mathsf{P}; (\overline{Y}_t)_{t\ge 0}\big).$$

For the last of these models, the (*NFLVR*) and the completeness with the stochastic integrals up to infinity are obviously equivalent to the (*NFLVR*) and the completeness with the improper stochastic integrals. Now, the desired result follows from Propositions 5.2, 5.4. □

`Remarks.` (i) Theorem 5.5 shows that the existence of an equivalent σ-martingale measure can be guaranteed by the condition weaker than (*NFLVR*).

(ii) The (*NFLVR*) condition and the completeness condition with the stochastic integrals up to infinity are more convenient than the original ones since we have a predictable description for the integrability up to infinity, while there seems to be no such a description for the improper integrability. □

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