A new polynomial continued fraction for $\exp(\pi)$

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The number $\exp(\pi)$

• By means of functional interpolation theory over $\mathbb{Z}[i]$, Gel'fond proved in 1929 that $e^\pi \notin \overline{\mathbb{Q}}$.

• Using elliptic functions, Chudnovky proved in 1978 that π and $\Gamma(1/4)$ are algebraically independent over $\overline{\mathbb{Q}}$.

• As a corollary of his theorem on values of modular functions, Nesterenko proved in 1996 that π, $\Gamma(1/4)$ and e^π are algebraically independent over $\overline{\mathbb{Q}}$.

• Building upon Gel'fond's method, Koksma and Popken (1932) proved that, for all $\alpha \in \overline{\mathbb{Q}}^*$ with naive height $H \geq 3$ and degree D, and all $\varepsilon > 0$,

$$|e^\pi - \alpha| \gg_{D,\varepsilon} \exp\left(-(4+\varepsilon)\log^2(H)/\log\log(H)\right).$$

• Baker (1973): there exist two constants $c_0, c_1 > 0$ such that for all $(p, q) \in \mathbb{Z} \times \mathbb{N}$, $q \geq 2$,

$$\left|e^\pi - \frac{p}{q}\right| \geq \frac{c_0}{q^{c_1 \log\log(q)}}.$$

• Baker's measure seems to be the best currently known.It is not known whether e^π is a Liouville number or not.

Polynomial continued fractions

• A continued fraction representation for $\xi \in \mathbb{R}$ with *eventually polynomial elements* (PCF) is

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}} = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots$$

where for each $n \geq 0$, $a_n, b_n \in \overline{\mathbb{Q}}$ and for $n \geq N_0$, $a_n = A(n)$, $b_n = B(n)$ for $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ and an integer N_0.

• a_n's and b_n's are the *elements* of the continued fraction and the *convergents* are

$$\frac{p_n}{q_n} := b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n},$$

where $p_0 := b_0$, $q_0 := 1$ and (by convention) $p_{-1} := 1$, $q_{-1} := 0$.

• $(p_n)_{n \geq -1}$ and $(q_n)_{n \geq -1}$ satisfy the linear recurrence

$$U_n = b_n U_{n-1} + a_n U_{n-2}, \quad n \geq 1.$$

• We shall be interested in the case where $a_n, b_n \in \mathbb{Q}$ and $A(x), B(x) \in \mathbb{Q}[x]$.

Examples of PCFs

$$\sqrt{2} = 1 + \frac{1|}{|2} + \frac{1|}{|2} + \cdots + \frac{1|}{|2} + \cdots$$

$$e = 1 + \frac{1|}{|1} + \frac{1|}{|2} + \frac{2|}{|3} + \cdots + \frac{n|}{|n+1} + \cdots$$

$$\pi = 3 + \frac{1^2|}{|6} + \frac{3^2|}{|6} + \cdots + \frac{(2n+1)^2|}{|6} + \cdots$$

$$\zeta(3) = \frac{6|}{|5} - \frac{1^6|}{|117} - \frac{2^6|}{|535} - \cdots - \frac{n^6|}{|34n^3 + 51n^2 + 27n + 5} - \cdots$$

$$\int_0^\infty \frac{e^{-x}}{1+x}\,dx = \frac{1|}{|2} - \frac{1|}{|4} - \frac{4|}{|6} - \cdots - \frac{n^2|}{|2(n+1)} + \cdots$$

$$-\frac{\Gamma(1/3)^3}{2\pi\sqrt{3}} = 1 + \frac{104|}{|s(0)} + \frac{r(0)t(1)|}{|s(1)} + \cdots + \frac{r(n)t(n+1)|}{|s(n+1)} + \cdots$$

where $r(n), s(n), t(n) \in \mathbb{Z}[n]$ are each of degree 4.

$\pi\coth(\pi)$ also has a PCF representation with rational elements eventually in $\mathbb{Q}[n]$.

• Many PCFs can be deduced from Euler's identity

$$\sum_{n=0}^{\infty}\Big(\prod_{j=0}^{n} a_j\Big) = \frac{a_0|}{|1} - \frac{a_1|}{|1+a_1} - \frac{a_2|}{|1+a_2} - \cdots - \frac{a_n|}{|1+a_n} - \cdots .$$

The sequence of partial quotients of the CF coincides with the sequence of partial sums of the series.

Exercice: find PCFs for e^{π} and $2^{\sqrt{2}}$ with algebraic elements eventually in $\overline{\mathbb{Q}}[n]$ (hint: use $(1+x)^s = \sum_{n\geq 0}\binom{s}{n}x^n$).

• Even with algebraic elements eventually in $\overline{\mathbb{Q}}[n]$, no PCFs are known for Euler's constant γ, $e+\pi$, $e\pi$ or π^e.

• It also seems to be unknown is all algebraic numbers can be represented as PCFs with rational elements eventually in $\mathbb{Q}[n]$.

• I will now focus on the number e^{π}.

Theorem 1

Set $a(x) := 2(x+1)^2 + 8$ *and* $b(x) := 2x + 3$.

The number e^π *is representable by the two (inequivalent) PCFs:*

$$e^\pi = 1 + \frac{6|}{|1} - \frac{560|}{|800} - \frac{c(2)|}{|d(2)} - \frac{c(3)|}{|d(3)} - \cdots - \frac{c(n)|}{|d(n)} - \cdots \tag{1}$$

$$= -3 + \frac{200|}{|4} + \frac{3744|}{|1064} - \frac{e(2)|}{|f(2)} - \frac{e(3)|}{|f(3)} - \cdots - \frac{e(n)|}{|f(n)} - \cdots \tag{2}$$

where

$$\begin{aligned}
c(x) &:= 4a(2x-2)a(2x-1)b(2x-4)b(2x),\\
d(x) &:= 2b(2x-2)a(2x) + 4b(2x-2)b(2x-1)b(2x) + 2a(2x-1)b(2x),\\
e(x) &:= a(2x-1)a(2x)b(2x-3)b(2x+1),\\
f(x) &:= b(2x-1)a(2x+1) + 2b(2x-1)b(2x)b(2x+1) + a(2x)b(2x+1).
\end{aligned}$$

• Inequivalent means that the CFs (1) and (2) do not have the same sequences of convergents $(\widehat{p}_n/\widehat{q}_n)_n$ and $(\widetilde{p}_n/\widetilde{q}_n)_n$.

• We have

$$\lim_{n\to+\infty}\left|e^\pi-\frac{\widehat{p}_n}{\widehat{q}_n}\right|^{1/n}=\lim_{n\to+\infty}\left|e^\pi-\frac{\widetilde{p}_n}{\widetilde{q}_n}\right|^{1/n}=\frac{1}{4}(2-\sqrt{2})^4$$

and

$$\lim_{n\to+\infty}|\widehat{q}_n/n!^3|^{1/n}=32(2+\sqrt{2})^2,\ \lim_{n\to+\infty}|\widetilde{q}_n/n!^3|^{1/n}=16(2+\sqrt{2})^2.$$

• These sequences are not good enough to prove the irrationality of e^π nor that e^π is not à Liouville number.

Proposition 1

Let $\alpha \in \mathbb{C}^*$ *be such that* $|\arg(\alpha)| < \pi$ *and* $\beta \in \mathbb{C} \setminus \mathbb{Z}$. *Set* $A(x) := (x+1)^2 - \beta^2$ *and* $B(x) := (\alpha+1)(2x+3)$. *Then*

$$\alpha^\beta = 1 + \frac{2\beta(\alpha-1)|}{|\alpha+\beta+1-\alpha\beta} - \frac{(\alpha-1)^2A(0)|}{|B(0)} - \frac{(\alpha-1)^2A(1)|}{|B(1)} - \frac{(\alpha-1)^2A(2)|}{|B(2)} - \cdots - \frac{(\alpha-1)^2A(n)|}{|B(n)} - \cdots. \quad (3)$$

I found and proved (3) before finding that it seems to be due to Euler!

- Take $\alpha = e^{i\pi/2}$ and $\beta = 2i$ in (3):

$$e^\pi = 1 + \frac{4(i+1)|}{|-(i+1)} + \frac{ia(0)|}{|(i+1)b(0)} + \frac{ia(1)|}{|(i+1)b(1)} + \cdots + \frac{ia(n)|}{|(i+1)b(n)} + \cdots \quad (4)$$

where $a(x) := 2(x+1)^2 + 8$ and $b(x) := 2x+3$.

- Eq. (4) is a PCF for e^π but with coefficients in $\mathbb{Q}[i]$, not in $\mathbb{Q}$.

• Nice unexpected surprise: all occurences of i can be eliminated in (4) :

$$\begin{aligned} e^{\pi} &= 1 + \frac{4 \,|}{|\, -1} + \frac{a(0) \,|}{|\, 2b(0)} + \frac{a(1) \,|}{|\, b(1)} + \cdots + \frac{a(2n) \,|}{|\, 2b(2n)} + \frac{a(2n+1) \,|}{|\, b(2n+1)} + \cdots \qquad (5) \\ &= 1 - \frac{4 \,|}{|\, 1} - \frac{10 \,|}{|\, 6} + \frac{16 \,|}{|\, 5} + \frac{26 \,|}{|\, 14} + \frac{40 \,|}{|\, 9} + \frac{58 \,|}{|\, 22} + \frac{80 \,|}{|\, 13} + \frac{106 \,|}{|\, 30} + \frac{136 \,|}{|\, 17} + \frac{170 \,|}{|\, 38} + \cdots \end{aligned}$$

• But (5) **is not** a PCF because the polynomials a and b are repeated "twice in a row".

• Take $\alpha = 2$ and $\beta = \sqrt{2}$ in (3):

$$2^{\sqrt{2}} = 1 - \frac{2\sqrt{2} \,|}{|\, \sqrt{2} - 3} - \frac{-1 \,|}{|\, 9} - \frac{2 \,|}{|\, 15} - \frac{7 \,|}{|\, 21} - \cdots - \frac{n^2 + 2n - 1 \,|}{|\, 3(2n+3)} - \cdots .$$

Is it possible to "eliminate" the two occurences of $\sqrt{2}$ in this PCF?

- To obtain (1) and (2), take the "odd" resp. "even" part of (5):
- Odd part

$$\frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \cdots = \frac{a_1}{b_1} - \frac{a_1 a_2 b_3 / b_1|}{|b_1(a_3 + b_2 b_3) + a_2 b_3} - \frac{a_3 a_4 b_5 b_1|}{|b_3(a_5 + b_4 b_5) + a_4 b_5} - \frac{a_5 a_6 b_7 b_3|}{|b_5(a_7 + b_6 b_7) + a_6 b_7} - \cdots$$

- Even part:

$$\frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \cdots = \frac{b_2 a_1|}{|b_2 b_1 + a_2} - \frac{a_2 a_3 b_4|}{|b_2(a_4 + b_3 b_4) + a_3 b_4} - \frac{a_4 a_5 b_6 b_2|}{|b_4(a_6 + b_5 b_6) + a_5 b_6} - \frac{a_6 a_7 b_8 b_4|}{|b_6(a_8 + b_7 b_8) + a_7 b_8} - \cdots$$

A generalization to $\exp(r\pi)$ for $r \in \mathbb{Q}$

- For every $r \in \mathbb{Q}$,

$$e^{r\pi} = 1 + \frac{4r}{|1-2r} + \frac{a_r(0)|}{|2b(0)} + \frac{a_r(1)|}{|b(1)} + \cdots + \frac{a_r(2n)|}{|2b(2n)} + \frac{a_r(2n+1)|}{|b(2n+1)} + \cdots$$

where $a_r(x) := 2(x+1)^2 + 8r^2$ and $b(x) := 2x+3$.

- If $r = 1/2$, the above result reads

$$e^{\pi/2} = 1 + \frac{4b(0)}{a_r(0)} + \frac{2a_r(1)/a_r(0)|}{|b(1)} + \frac{a_r(2)|}{|2b(2)} + \frac{a_r(3)|}{|b(3)} + \cdots + \frac{a_r(2n+1)|}{|b(2n+1)} + \frac{a_r(2n+2)|}{|2b(2n+2)} + \cdots$$

- PCFs for $e^{r\pi}$ can be deduced as well.

- If $r^2 \in \mathbb{Q}$, observe that $a_r(x) \in \mathbb{Q}[x]$. We thus obtain a PCF for $e^{r\pi}$ with all elements in $\mathbb{Q}$ except (possibly) the elements $4r$ and $1-2r$.

A proof of Euler's identity

- For $m, n \geq 0$, set

$$I(m,n) := \frac{m!n!}{2i\pi} \int_{\mathcal{C}} \frac{\alpha^z \, dz}{\left(\prod_{j=0}^{m}(z-j)\right)\left(\prod_{j=0}^{n}(z-j-\beta)\right)}$$

where $\mathcal{C}$ is any closed direct curve surrounding the poles of the integrand.

- $I(m,n)$ is a variation of integrals in interpolation theory. Here, the interpolating sets are $\mathbb{N}$ and $\mathbb{N}+\beta$.

- Generalisations of $I(m,n)$ enabled Gel'fond , resp. Kuzmin, to prove the transcendence of e^{π}, resp. $2^{\sqrt{2}}$, by interpolating $e^{\pi z}$ and 2^z on $\mathbb{Z}[i]$, resp. $\mathbb{Z}[\sqrt{2}]$.

- Taking $n = 0$ and allowing multiplicities for the poles at $0, 1, \ldots, m$, the case $\alpha = e$ leads to a proof of the transcendence of e.

Proposition 2

Let $\alpha \in \mathbb{C}^*$ *such that* $|\arg(\alpha)| < \pi$ *and* $\beta \in \mathbb{C} \setminus \mathbb{Z}$.

- *For* $m, n \geq 0$, *we have* $I(m,n) = Q(m,n)\alpha^\beta - P(m,n)$, *where*

$$Q(m,n) := (-1)^n \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} m!}{\prod_{j=0}^{m}(k-j+\beta)} \alpha^k$$

and

$$P(m,n) := (-1)^{m+1} \sum_{k=0}^{m} \frac{(-1)^k \binom{m}{k} n!}{\prod_{j=0}^{n}(k-j-\beta)} \alpha^k.$$

- *The three sequences* $P(n,n)$, $Q(n,n)$ *and* $I(n,n)$ *are solutions of the linear recurrence of order 2:*

$$((n+2)^2 - \beta^2)U_{n+2} - (\alpha+1)(n+2)(2n+3)U_{n+1} + (\alpha-1)^2(n+1)(n+2)U_n = 0.$$

PCF (3) in Proposition 1 follows.