Siegel's problem for E-functions

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E- and G-functions (Siegel, 1929)

Definition 1

A power series $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$ is an E-function if

(i) $F(z)$ is solution of a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

(ii) There exists $C > 0$ such that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $n \geq 0$, $|\sigma(a_n)| \leq C^{n+1}$.

(iii) There exists $D > 0$ and a sequence of positive integers d_n, with $d_n \leq D^{n+1}$, such that $d_n a_m$ are algebraic integers for all $m \leq n$.

By (i), the a_n's all lie in a Galoisian number field $\mathbb{K}$. There are only finitely many Galois conjugates $\sigma(a_n)$ of a_n to consider in (ii).

E-functions are entire functions. They form a ring stable under $\frac{d}{dz}$ and $\int_0^z$. If $F(z)$ is an E-function and $\alpha \in \overline{\mathbb{Q}}$, then $F(\alpha z)$ is an E-function.

A power series $\sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is a G-function if $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an E-function.

Examples

E-functions: polynomials in $\overline{\mathbb{Q}}[z]$,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad L(z) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \right) \frac{z^n}{n!},$$

$$H(z) := \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k} \right) \frac{z^n}{n!}, \quad J_0(z) := \sum_{n=0}^{\infty} \frac{(iz/2)^{2n}}{n!^2}.$$

G-functions: algebraic functions over $\overline{\mathbb{Q}}(z)$ regular at 0, (multi)polylogarithms

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} \qquad (s \in \mathbb{Z}),$$

$$\sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}} \qquad (s_1, s_2, \ldots, s_k \in \mathbb{Z}),$$

$$\frac{1}{\pi} \int_0^1 \frac{\sqrt{x(1-x)}}{1-zx} dx.$$

The intersection of both classes of series is reduced to $\overline{\mathbb{Q}}[z]$.

Hypergeometric series

Set $(x)_m := x(x+1)\cdots(x+m-1)$.

Siegel: the "hypergeometric" series

$$
{}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z^{q-p+1}\right] := \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{n!(b_1)_n\cdots(b_q)_n}z^{n(q-p+1)},
$$

is an E-function when $q \geq p \geq 1$, $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ for all j.

$L(z)$ and $H(z)$ are not of ${}_pF_q(z^{q-p+1})$ type but

$$
\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}\right)\frac{z^n}{n!} = e^{(3-2\sqrt{2})z}\cdot{}_1F_1\left[\begin{matrix}1/2\\ 1\end{matrix};4\sqrt{2}z\right],
$$

$$
\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\frac{z^n}{n!} = ze^z\cdot{}_2F_2\left[\begin{matrix}1,1\\ 2,2\end{matrix};-z\right].
$$

Observation: Every polynomial with coefficients in $\overline{\mathbb{Q}}$ in series ${}_pF_q[a_1,\dots,a_p;b_1,\dots,b_q;\lambda z^{q-p+1}]$, with $q \geq p \geq 1$, $a_j, b_j \in \mathbb{Q}$ and $\lambda \in \overline{\mathbb{Q}}$, is an E-function. (The integers p and q can take different values for each series involved in the polynomial.)

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Observation: Every polynomial with coefficients in $\overline{\mathbb{Q}}$ in series ${}_pF_q[a_1,\ldots,a_p;b_1,\ldots,b_q;\lambda z^{q-p+1}]$, with $q \geq p \geq 1$, $a_j, b_j \in \mathbb{Q}$ and $\lambda \in \overline{\mathbb{Q}}$, is an E-function. (The integers p and q can take different values for each series involved in the polynomial.)

Siegel's question

Question 1 (Siegel, 1949)

Is it possible to write every E-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of series of the form ${}_pF_q[a_1,\ldots,a_p;b_1,\ldots,b_q;\lambda z^{q-p+1}]$, with $q\geq p\geq 1$, $a_j,b_j\in\mathbb{Q}$ and $\lambda\in\overline{\mathbb{Q}}$? (The integers p and q can take different values for each series involved in the polynomial.)

Observation: such a representation may not be unique. For instance

$$J_0(z) = {}_1F_2\left[\begin{matrix}1\\1,1\end{matrix};(iz/2)^2\right] = e^{-iz}\cdot {}_1F_1\left[\begin{matrix}1/2\\1\end{matrix};2iz\right].$$

Gorelov, 2004: the answer is yes if the E-function (even in the general sense) is solution of a non-zero linear differential equation of order ≤ 2 with coefficients in $\overline{\mathbb{Q}}(z)$.

In 2019, Fischler and myself gave a strong reason to believe that the answer was negative in general for differential order ≥ 3. The answer was then shown to be indeed negative by Fresán and Jossen in 2020, who produced an explicit counter-example.

In the rest of the talk, I will explain our 2019 result.

Rings of special values

G the ring of values taken at algebraic points by analytic continuations of G-functions. Algebraic numbers, $\Gamma(a/b)^b$ $(a, b \in \mathbb{N})$ and π are units of **G**.

H the ring generated by $\overline{\mathbb{Q}}$, $1/\pi$ and $\Gamma^{(n)}(r)$, $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $n \in \mathbb{N}$. Algebraic numbers and $\Gamma(r)$ $(r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0})$ are units of **H**.

S the **G**-module generated by $\Gamma^{(n)}(r)$, $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $n \in \mathbb{N}$. It is a ring.

G and **H** are subrings of **S**.

Proposition 1

(i) **H** *is generated by* $\overline{\mathbb{Q}}$, $1/\pi$ *and*

$$\begin{cases} \mathrm{Li}_s(e^{2i\pi r}) & s \in \mathbb{N}^*,\ r \in \mathbb{Q},\ (s, e^{2i\pi r}) \neq (1,1) \\ \log(q) & q \in \mathbb{N}^* \\ \Gamma(r) & r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0} \\ \gamma := -\Gamma'(1) & (\mathit{Euler's\ constant}) \end{cases}$$

(ii) **S** *is the* $\mathbf{G}[\gamma]$*-module generated by* $\Gamma(r)$, $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$.

Main result

Theorem 1 (Fischler-R., 2019)

At least one of the following statements is true:

(*i*) $\mathbf{G} \subset \mathbf{H}$;

(*ii*) *Siegel's question has a negative answer.*

(*i*) is very unlikely. It contradicts a conjecture on (exponential) periods that generalizes Grothendieck's periods conjecture.

If there exist $s \in \mathbb{N}^*$ and $\alpha \in \overline{\mathbb{Q}}$ such that $\mathrm{Li}_s(\alpha) \in \mathbf{G}$ is not in $\mathbf{H}$, then the E-function

$$\sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \frac{\alpha^k}{k^s} \right) \frac{z^n}{n!}$$

is a counter-example, of differential order at most $s+2$.

I will outline the proof of Theorem 1 when in Siegel's question we further assume that $p = q$. The series are then said to be confluent hypergeometric series.

The proof of the general case is based on the case $p = q$ together with more complicated arguments.

Asymptotic expansions in large sectors

Definition 2

Let $\theta \in \mathbb{R}$. We write

$$f(z) \sim \sum_{\rho\in\mathbb{C}} e^{\rho z} \sum_{\alpha\in\mathbb{C}} \sum_{i\in\mathbb{N}} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n}(\theta) \frac{\log(z)^i}{z^{n+\alpha}} \qquad (c_{\rho,\alpha,i,n}(\theta) \in \mathbb{C})$$

where the sums over ρ, α, i are finite, and we say that the RHS is the (asymptotic) expansion of f at ∞ in a large sector bisected by the direction θ, when:

There exist $\varepsilon, R, B, C > 0$ and certain functions $f_\rho(z)$ holomorphic in the sector

$$U := \left\{ z \in \mathbb{C},\ |z| \geq R,\ \theta - \frac{\pi}{2} - \varepsilon \leq \arg(z) \leq \theta + \frac{\pi}{2} + \varepsilon \right\},$$

such that $f(z) = \sum_\rho e^{\rho z} f_\rho(z)$ and

$$\left| f_\rho(z) - \sum_{\alpha\in\mathbb{C}} \sum_{i\in\mathbb{N}} \sum_{n=0}^{N-1} c_{\rho,\alpha,i,n}(\theta) \frac{\log(z)^i}{z^{n+\alpha}} \right| \leq \frac{C^N N!}{|z|^{N-B}}, \quad z \in U,\ N \geq 1.$$

If such an expansion exists in some large sector, it is unique in this sector.

Asymptotic expansions of E-functions

Theorem 2

(*i*) (André, 2000) *Let $f(z)$ be an E-function. There exists a finite set A such that, for any $\theta \in (-\pi, \pi) \setminus A$,*

$$f(z) \sim \sum_{\rho \in \overline{\mathbb{Q}}} e^{\rho z} \sum_{\alpha \in \mathbb{Q}} \sum_{i \in \mathbb{N}} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n}(\theta) \frac{\log(z)^i}{z^{n+\alpha}},$$

in a large sector bisected by the direction θ, where (Fischler-R., 2016) *the coefficients*

$$c_{\rho,\alpha,i,n}(\theta) \in \mathbf{S}.$$

(*ii*) (Fischler-R., 2019) *Let $\xi \in \mathbf{G}$. There exists an E-function $F(z)$ and a finite set S such that for any $\theta \in (-\pi, \pi) \setminus S$, ξ is one of the $c_{\rho,\alpha,i,n}(\theta)$ of the expansion of $F(z)$ in a large sector bisected by θ.*

Asymptotic expansions of confluent hypergeometric series

Theorem 3

Let $\theta \in (-\pi, \pi) \setminus \{0\}$, and $f(z)$ be a confluent hypergeometric series ${}_pF_p(z)$ with rational parameters. Then,

$$f(z) \sim \sum_{\rho \in \{0,1\}} e^{\rho z} \sum_{\alpha \in \mathbb{Q}} \sum_{i \in \mathbb{N}} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n}(\theta) \frac{\log(z)^i}{z^{n+\alpha}}$$

in a large sector bisected by the direction θ where (Fischler-R., 2019) *the coefficients*

$$c_{\rho,\alpha,i,n}(\theta) \in \mathbf{H}.$$

It is a consequence of Barnes and Wright's classical results, with refinements coming from the theory of Meijer's G-function.

Proof of Theorem 1 in the case $p = q$

Let $\xi \in \mathbf{G}$.

By Theorem 2(*ii*), there exist an E-function $F(z)$ and a finite set S such that for any $\theta \in (-\pi, \pi) \setminus S$, ξ is a coefficient of the expansion of $F(z)$ in a large sector bisected by θ.

Assume that Siegel's question has a positive answer (in the case $p = q$).

There exist ${}_pF_p(z)$ hypergeometric series $f_1, \dots, f_n$ with rational parameters, algebraic numbers $\lambda_1, \dots, \lambda_n$, and a polynomial $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$, such that

$$F(z) = P\big(f_1(\lambda_1 z), \dots, f_n(\lambda_n z)\big).$$

Choose $\theta \in (-\pi, \pi) \setminus S$ such that $\theta + \arg(\lambda_i) \notin \pi\mathbb{Z}$ for every i. By Theorem 3, the expansion of each $f_i(\lambda_i z)$ in a large sector bisected by θ has coefficients in $\mathbf{H}$. The same holds for $F(z)$ because $\mathbf{H}$ is a $\overline{\mathbb{Q}}$-algebra.

Such an expansion being unique, the coefficient ξ belongs to $\mathbf{H}$.