Weak convergence and statistics

Weak convergence of e.c.d.f.

Consider $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{X_i \le x}$. Let's prove the following result.

Theorem 1. *1. Let X_i be $R[0,1]$. Then*

$$\sqrt{n}(\hat{F}_n(x) - x) \xrightarrow{d} W_x^0$$

in $D[0,1]$.

2. Let $X_i \sim F$, where F is continuous d.f. Then

$$\sqrt{n}(\hat{F}_n(x) - F(x)) \xrightarrow{d} W_{F^{-1}}^0$$

in $D(\mathbb{R})$.

Proof. 1. Let $X_i \sim R[0,1]$.

(a) The finite-dimensional convergence of $Y_n(t)$ to W_t^0 has been proved at the end of the previous lection

(b) To prove the tightness of $\{\mathbf{P}(Y_n \in \cdot),\ n \ge 1\}$ we need to estimate $\mathbf{E}(Y_n(t) - Y_n(s))^2(Y_n(r) - Y_n(t))^2$, $s < t < r$, since

$$\mathbf{P}(|Y_n(t) - Y_n(s)| > \varepsilon, |Y_n(r) - Y_n(t)| > \varepsilon) \le \frac{\mathbf{E}(Y_n(t) - Y_n(s))^2(Y_n(r) - Y_n(t))^2}{\varepsilon^2}$$

Let $\xi_i = I_{X_i \in (s,t]} - (t-s)$, $\eta_i = I_{X_i \in (t,r]} - (r-t)$. Then

$$\mathbf{E}(Y_n(t) - Y_n(s))^2(Y_n(r) - Y_n(t))^2 = \frac{1}{n^2}\mathbf{E}(\xi_1 + \ldots + \xi_n)^2(\eta_1 + \ldots + \eta_n)^2 = \frac{1}{n^2}\sum_{i,j,k,l}\mathbf{E}\xi_i\xi_j\eta_k\eta_l.$$

If $i \notin \{j,k,l\}$, then $\mathbf{E}\xi_i\xi_j\eta_k\eta_l = \mathbf{E}\xi_i\mathbf{E}\xi_j\eta_k\eta_l = 0$. Therefore,

$$\sum_{i,j,k,l}\mathbf{E}\xi_i\xi_j\eta_k\eta_l = 4\sum_{i<j}\mathbf{E}\left(\xi_i\eta_i\right)\mathbf{E}\left(\xi_j\eta_j\right) + \sum_{i \ne j}\mathbf{E}\xi_i^2\mathbf{E}\eta_j^2 + \sum_i \mathbf{E}\xi_i^2\eta_i^2 =$$
$$2n(n-1)\left(\mathbf{E}\xi_1\eta_1\right)^2 + n(n-1)\mathbf{E}\xi_1^2\mathbf{E}\eta_1^2 + n\mathbf{E}\xi_1^2\eta_1^2.$$

Obviously,

$$\mathbf{E}\xi_1\eta_1 = \mathbf{E}I_{X_1 \in (s,t]}I_{X_1 \in (t,r]} - (t-s)(r-t) = -(t-s)(r-t),\ \left(\mathbf{E}\xi_1\eta_1\right)^2 \le (t-s)(r-t)$$
$$\mathbf{E}\xi_1^2 = \mathbf{E}I_{X_1 \in (s,t]} - (t-s)^2 = (t-s)(1-(t-s)),\ \mathbf{E}\eta_1^2 = (r-t)(1-(r-t)),$$
$$\mathbf{E}\xi_1^2\mathbf{E}\eta_1^2 \le (t-s)(r-t),\ \mathbf{E}\xi_1^2\eta_1^2 = (1-(t-s))^2(r-t)^2(t-s) +$$
$$(t-s)^2(1-(r-t))^2(r-t) + (t-s)^2(r-t)^2(1-r-s) \le 3(t-s)(r-t).$$

Thus,

$$\mathbf{E}(Y_n(t) - Y_n(s))^2(Y_n(r) - Y_n(t))^2 \le 6(t-s)(r-t) \le 6(r-s)^2.$$

2. Suppose that $X_i \sim F(x)$, where F is continuous. Then

$$\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} I_{X_i \le t} = \frac{1}{n}\sum_{i=1}^{n} I_{Y_i \le F^{-1}(t)}, F(t) = \mathbf{P}(X \le t) = \mathbf{P}(R \le F^{-1}(t)),$$

where $R \sim R[0,1]$. Therefore, $Y_n(t) = \tilde{Y}_n(F(t))$, where $\tilde{Y}_n(t)$ is defined as in Part 1). Consider the functional $f(G) = G(F(t))$. If $G_n \to G$ in $(D[0,1], \rho_2)$, where $G \in C[0,1]$, then $f(G_n) \to f(G)$. Therefore

$$Y_n(t) \xrightarrow{d} W_{F(t)}^0,\ n \to \infty.$$

□

It's interesting that the theorem is true for discontinuous F too.

Theorem 2 (Skorohod Representation Theorem). *Suppose that $\mathbf{P}_n \xrightarrow{d} \mathbf{P}$, where $\mathbf{P}$ are measures on $S, \mathcal{S}$, where S is a separable space, $\mathcal{S} = \mathcal{B}(S)$. Then there exist random elements X_n, X, defined on a common probability space $(\Omega, \mathcal{F}, \tilde{\mathbf{P}})$, such that $\tilde{\mathbf{P}}(X_n \in A) = \mathbf{P}_n(A)$, $\tilde{\mathbf{P}}(X \in A) = \mathbf{P}(A)$, $X_n \to X$ a.s.*

We left it without the proof.

Delta method 2.0

Theorem 3 (Delta Method 2.0). *Let X_n, Y be random elements, $X_n : \Omega \to S$, where S is a separable metric space with Borel σ-algebra, $F \in S$, $r_n \to \infty$ and suppose that $r_n(X_n - F) \xrightarrow{d} Y$. Let f be an Hadamard differentiable functional. Then*

$$r_n(f(X_n) - f(F)) \xrightarrow{d} f'_F(Y),$$

where $f'_F(Y)$ is a Gateuax derivative of f at F in the direction Y.

Proof. Due to the Representation Theorem there exist $\tilde{X}_n \stackrel{d}{=} X_n$, $\tilde{Y} \stackrel{d}{=} Y$:

$$r_n(\tilde{X}_n - F) \xrightarrow{a.s.} \tilde{Y}.$$

Then

$$r_n(f(\tilde{X}_n) - f(F)) = \frac{f(F + r_n(\tilde{X}_n - F)/r_n) - f(F)}{r_n^{-1}} \xrightarrow{a.s} f'_F(\tilde{Y})$$

due to the definition of Hadamard differentiability. Therefore,

$$r_n(f(X_n) - f(F)) \xrightarrow{d} Y$$

□

Particularly,

$$\sqrt{n}(f(\hat{F}_n) - f(F) \xrightarrow{d} f'_F(W^0_{F(t)}).$$

Similarly,

$$\sqrt{n}(f(\hat{F}_n, \hat{G}_m) - f(F, G)) \xrightarrow{d} f'_{F,G}(W^0_{1,F(t)}, \sqrt{\alpha} W^0_{2,G(t)}),$$

as $n, m \to \infty$, $n/m \to \alpha \in (0, 1)$, W^0_1, W^0_2 are independent Brownian Bridges.

Delta Method and Delta Method 2.0

This subsection is not necessary for the exams
Why $f'_F(W^0_{F(t)}) \sim \mathcal{N}(0, \sigma^2(F))$? Nonformally,

$$f'_F(W^0_{F(t)}) = \int_{\mathbb{R}} f'_F(\delta_x) dW^0_{F(x)}.$$

An integral above is a limit of

$$Y_m = \sum_{i=1}^{m} f'_F(\delta_{x_i})(W^0_{F(x_{i+1})} - W^0_{F(x_i)}),$$

where $x_i = (i-1)/m$. Then $\mathbf{E}Y_m = 0$ and

$$cov(Y_m, Y_m) = \sum_{i=1}^{m}\sum_{j=1}^{m} f'_F(\delta_{x_i}) f'_F(\delta_{x_j}) \mathbf{E}(\Delta_i W_F - \Delta_i\ F W_1)(\Delta_j W_F - \Delta_j\ F W_1) = \sum_{i,j} I_F(x_i) I_F(x_j) a_{i,j},$$

where $\Delta_i f = f(x_{i+1}) - f(x_i)$. Since

$$a_{i,i} = \mathbf{E}(\Delta_i W_F - \Delta_i F\ W_1)^2 = \Delta_i F - (\Delta_i F)^2, a_{i,j} = \mathbf{E}(\Delta_i W_F - \Delta_i F\ W_1)(\Delta_j W_F - \Delta_j F\ W_1) = -(\Delta_i F)(\Delta_j F),$$

we have

$$cov(Y_m, Y_m) = \left(\sum_{i=1}^{m}(f'_F(\delta_{x_i}))^2\Delta_i F\right) - \left(\sum_{i=1}^{m} f'_F(\delta_{x_i})\Delta_i F\right)^2 \to \int_{\mathbb{R}}(f'_F(\delta_x))^2 dF(x) - \left(\int_{\mathbb{R}} f'_F(\delta_x) dF(x)\right)^2 =$$
$$\int_{\mathbb{R}}(f'_F(\delta_x) - f'(F)(F))^2 dF(x) = \int_{\mathbb{R}}(f'_F(\delta_x - F(x)))^2 dF(x) = \int_{\mathbb{R}} I_F(x)^2 dF(x) = \sigma^2(F).$$

High Order Derivatives

Theorem 4 (Generalized Delta Method)**.** *Let S be a separable metric space, X_n, Y — random elements in S, $F \in S$, $r_n \to \infty$ and suppose that $r_n(X_n - F) \xrightarrow{d} Y$. Let f be a functional, satisfying the following conditions:*
1) $f(F + t(G - F))^{(j)}_{t=0} = 0$ as $j = 1, ..., k$ for every G,
2) $f(F + tG_n)^{(k+1)}_{t=t_n} \to I_{k,F}(G) = f(F + tG)^{(k+1)}_{t=0}$ as $G_n \to G$, $t_n \to 0$.
Then

$$r_n^{k+1}(f(X_n) - f(F)) \xrightarrow{d} \frac{I_{k,F}(Y)}{(k+1)!}.$$

Proof. Consider $\tilde{X}_n \stackrel{d}{=} X_n$, $\tilde{Y} \stackrel{d}{=} Y$:

$$r_n(\tilde{X}_n - F) \to \tilde{Y}, \quad \text{a.s.}$$

Then

$$r_n^{k+1}(f(\tilde{X}_n) - f(F)) = r_n^{k+1} \cdot \frac{r_n^{-k-1}}{(k+1)!} f(F + t(\tilde{X}_n - F))^{(k+1)}_{t=\xi_n} \to \frac{I_{k,F}(\tilde{Y})}{(k+1)!}$$

a.s. as $n \to \infty$, where $\xi_n \in [0, r_n^{-1}]$. Therefore,

$$r_n^{k+1}(f(\tilde{X}_n) - f(F)) \xrightarrow{d} \frac{I_{k,F}(Y)}{(k+1)!}$$

□