Глава 2

Lection 2. Consistency and Asymptotically Normality

Example 1. We show that the median $f_2(F) = F^{-1}(1/2)$ is not a weakly continuous functional. Does $f_2(\widehat{F}_n)$ converge to $f_2(F)$? Let F be a c.d.f. of $X \sim Bern(1/2)$. Then

$$\widehat{F}_n^{-1}\left(\frac{1}{2}\right) = \begin{cases} 1, & \#\{i : X_i = 1\} > n/2, \\ 0, & \#\{i : X_i = 1\} \le n/2. \end{cases}, \quad F^{-1}\left(\frac{1}{2}\right) = 0.$$

We see that

$$\mathbf{P}\left(\left|f_2(\widehat{F}_n) - f_2(F)\right| > 1/2\right) = \mathbf{P}(S_n \ge n/2) \to 1 - \Phi(0) = \frac{1}{2}.$$

Therefore, $f_2(\widehat{F}_n)$ is not consistent for $f_2(F)$.

However, if $F(x_{1/2} + \varepsilon) > 1/2$ for every $\varepsilon > 0$ then f_2 is weakly continuous and $f_2(\widehat{F}_n)$ is consistent. Really,
a) Suppose that $F(x_{1/2}) = 1/2$. Then $F(x_{1/2} - \varepsilon) < 1/2$ or every $\varepsilon > 0$ and

$$F_n(x_{1/2} + \varepsilon) > 1/2, \quad \widehat{F}_n(x_{1/2} - \varepsilon) < 1/2,$$

for all n large enough, where $F_n \xrightarrow{d} F$, $n \to \infty$. So $F_n^{-1}(1/2) \in [x_{1/2} - \varepsilon, x_{1/2} + \varepsilon]$ and $F_n^{-1}(1/2) \to x_{1/2}$ as $n \to \infty$.
b) Suppose that $F(x_{1/2}) > 1/2$. Then $F(x_{1/2} - \varepsilon) < 1/2$ for every $\varepsilon > 0$ and

$$F_n(x_{1/2}) > 1/2, \quad F_n(x_{1/2} - 0) < 1/2$$

for all n large enough n. Similarly, $f(F_n) \to f(F)$ as $n \to \infty$.
Therefore,

$$f(\widehat{F}_n) = \widehat{F}_n^{-1}(1/2) = \begin{cases} X_{(k)}, & n = 2k, \\ X_{(k+1)}, & n = 2k + 1. \end{cases}$$

is consistent for $x_{1/2}$. Similarly,

$$MED = \begin{cases} X_{(k)}, & n = 2k, \\ X_{(k+1)}, & n = 2k + 1. \end{cases}$$

is consistent for $f_2(F) = (F^{-1}(1/2) + F^{-1}(1/2 + 0))/2$ as $F(x_{1/2} + \varepsilon) > F(x_{1/2} - 0)$ for every $\varepsilon > 0$.

Delta Method

As we see in previous section an estimator $f(\hat{F}_n)$ is consistent for $f(F)$ as the functional f is weakly continuous. Can we find a sufficient condition for asymptotic normality of $f(\hat{F}_n)$?

Definition 1. Let $\mathcal{D}$ be linear space generated by the set of all distribution functions, $f : \mathcal{F} \to \mathbb{R}^k$ is a functional. The *Gateaux derivative* of f at point F in direction $D \in \mathcal{D}$ is defined by

$$L_F(D) := \lim_{\varepsilon \to 0} \frac{f(F + \varepsilon D) - f(F)}{\varepsilon}.$$

Definition 2. The functional f is called *Hadamard differentiable* at F if there exists $L_F : \mathcal{D} \to \mathbb{R}^k$ such that for any $\varepsilon_n \to 0$ and $\{D, D_1, ...\} \subset \mathcal{D} : \sup_x |D_n(x) - D(x)| \to 0, n \to \infty,\ F_n + \varepsilon n D_n \in \mathcal{F}$

$$\lim_{n \to \infty} \left(\frac{f(F + \varepsilon_n D_n) - f(F)}{\varepsilon_n} - L_F(D_n) \right) = 0$$

Definition 3. *An influence function* of the functional f is defined by

$$I_F(x) := L_F(\delta_x - F) = \lim_{\varepsilon \to 0} \frac{f((1-\varepsilon)F + \varepsilon \delta_x) - f(F)}{\varepsilon},$$

where $\delta_x(u) = I_{u \geq x}$ is the c.d.f. of x.

Theorem 1. *Let f be Hadamard differentiable functional on $\mathcal{F}$. Then $f(\hat{F}_n)$ is an asymptotical normal estimator for $f(F)$ with $\sigma^2(F) = \int_{\infty}^{\infty} I_F(x)^2 dF(x)$.*

Example 2. Consider $f_1(F) = \int_{\mathbb{R}} a(x)dF(x) = \mathbf{E}_F a(X)$, where a is continuous function. Then

$$L_F(D) = \lim_{\varepsilon \to 0} \left(\frac{f(F + \varepsilon D) - f(F)}{\varepsilon} \right) = \int_{\mathbb{R}} a(x) dD(x).$$

Therefore,

$$\lim_{n \to \infty} \left(\frac{f(F + \varepsilon_n D_n) - f(F)}{\varepsilon_n} - L_F(D_n) \right) = \int_{\mathbb{R}} a(x) dD_n(x) - \int_{\mathbb{R}} a(x) dD_n(x) = 0$$

and f_1 is Hadamard differentiable. The influence function I_f is equal to

$$\lim_{\varepsilon \to 0} \frac{(1-\varepsilon) \int_{\mathbb{R}} a(u)dF(u) + \varepsilon \int_{\mathbb{R}} a(u) d\delta_x(u) - \int_{\mathbb{R}} a(u) dF(u)}{\varepsilon} =$$
$$\int_{\mathbb{R}} a(u) d\delta_x(u) - \int_{\mathbb{R}} a(u) dF(u) = a(x) - \mathbf{E}_F a(X).$$

Therefore,

$$\sigma^2(F) = \int_{\mathbb{R}} (a(x) - \mathbf{E}_F a(X))^2 dF(x) = \mathbf{D}_F a(X).$$

So, the Central Limit Theorem is a particular case of the Functional Delta Method.