Graphs, random graphs, and their extremal characteristics

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Trivially, $\chi(G) \geqslant \omega(G)$ and $\chi(G) \geqslant |V|/\alpha(G)$. Which bound is better?

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Theorem

Let p be a constant or a function tending to zero and bounded from below by a value $\frac{c}{n}$, where $c > 1$. Let $d = \frac{1}{1-p}$. Then w.h.p. $\alpha(G(n,p)) \sim 2\log_d(np)$ and $\chi(G(n,p)) \sim n/2\log_d(np)$. In particular, if $p = 1/2$, then w.h.p. $\alpha(G(n,p)) \sim 2\log_2 n$ and so the same is true for ω, that is, for almost all graphs the bound $|V|/\alpha$ is much better than the one by ω.

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A general random subgraph

Let $n \in \mathbb{N}$, $p \in [0,1]$, $G_n = (V_n, E_n)$ — an arbitrary sequence of graphs. $G_{n,p}$ is obtained from G_n by keeping independently edges of G_n, each with probability p.

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What can be said about $\alpha(G_{n,p})$ and $\chi(G_{n,p})$?

A special case

Main definition

Let $r, s, n \in \mathbb{N}$, $s < r < n$, and let $G(n, r, s) = (V, E)$, where

$$V = \{\mathbf{x} = (x_1, \ldots, x_n) : \; x_i \in \{0, 1\}, \; x_1 + \ldots + x_n = r\},$$

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Equivalent definition

Let $r, s, n \in \mathbb{N}$, $s < r < n$. Let $[n]$ be an n-element set, and let $G(n, r, s) = (V, E)$, where

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Again, what can be said about $\alpha(G_p(n, r, s))$ and $\chi(G_p(n, r, s))$?

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- **Combinatorial geometry**: $G(n, r, s)$ is a "**distance**" graph, i.e., its edges are of the same length $\sqrt{2(r-s)}$. The chromatic number $\chi(G(n, r, s))$ provides important bounds in the Nelson–Hadwiger problems of space coloring as well as in the Borsuk problem of partitioning sets in spaces into parts of smaller diameter.

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- Constructive bounds for Ramsey numbers.

Random subgraphs of $G(n, r, s)$: independence numbers

Theorem (Frankl, Füredi, 1985)

Let r, s be fixed as $n \to \infty$.

- If $r \leqslant 2s + 1$, then $\alpha(G(n, r, s)) = \Theta(n^s)$.

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A simple construction: Fix $1, \dots, s + 1$ and take all the vertices (subsets) containing them. Obviously it's an independent set of size $\binom{n-s-1}{r-s-1} = \Theta\left(n^{r-s-1}\right)$.

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Fix a real number $\varepsilon > 0$ and let $r = r(n)$ be a natural number such that $2 \leqslant r(n) = o(n^{1/3})$. Let $p_c(n,r) = ((r+1)\log n - r\log r)/\binom{n-1}{r-1}$. As $n \to \infty$,

$$\mathbb{P}\left(\alpha(G_p(n,r,0)) = \alpha(G(n,r,0)) = \binom{n-1}{r-1}\right) \to \begin{cases} 1 & \text{if } p \geqslant (1+\varepsilon)p_c(n,r) \\ 0 & \text{if } p \leqslant (1-\varepsilon)p_c(n,r). \end{cases}$$

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No other cases of strong stability are known.

Random subgraphs of $G(n,r,s)$: independence numbers for $r \leqslant 2s+1$

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Theorem (Bogoliubskiy, Gusev, Pyaderkin, A.M., 2013–2016)

Let r, s be fixed as $n \to \infty$. If $r \leqslant 2s+1$, then w.h.p.
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There are only two more cases where the Θ notation is replaced by the $\sim$ one.

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Random subgraphs of $G(n, r, s)$: chromatic numbers

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Many improvements by Kupavskii and by Alishahi and Hajiabolhassan.

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If $r \geqslant 3$, then w.h.p.

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A general result

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Theorem (A.M., 2017)

Let $G_n = (V_n, E_n)$, $n \in \mathbb{N}$, be a sequence of graphs. Let $N_n = |V_n|$, $\alpha_n = \alpha(G_n)$. Let γ_n be the maximum number of vertices of G_n that are non-adjacent to both vertices of a given edge. Assume that the quantities N_n, α_n, γ_n are monotone increasing to infinity and there exists a function β_n such that

1. $\beta_n > \gamma_n$ and $\beta_n = o(\alpha_n)$;
2. $\log_2 N_n = o\left(\frac{\alpha_n}{\beta_n}\right)$;
3. $\log_2 N_n = o\left(\beta_n - \gamma_n\right)$.

Then w.h.p. $\alpha(G_n, 1/2) \sim \alpha(G_n)$.