Typical metric properties of n-vertex graphs of given diameter

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When studying a given class of graphs admitting a notion of dimension, that is, the measure of their quantity (often the dimension of a graph is understood as the number of its vertices, of course, there are other approaches), questions of an asymptotic nature naturally arise.

Under the asymptotic investigation of the class Ω_n of graphs of dimension n, special attention is drawn to the topics around the following three questions.

In this definition, we consider graphs as combinatorial objects admitting the concept of dimension. And by *dimension* we mean a function that assigns to each object of the class under consideration a natural number that reflects the quantitative characteristic of this combinatorial object. In our case the dimension of a graph is understood as the number of its vertices.

Asymptotic investigation of the class Ω_n

- The first is a calculation of the asymptotically exact value of the number of such graphs of a given dimension (or to obtain good estimates for it). This makes it possible, with the specified accuracy, to easily calculate, as a rule, the difficult-to-calculate number $|\Omega_n|$.
- The second question is an extraction or a construction of a subclass of typical graphs $\Omega_n^* \subseteq \Omega_n$ for the given class Ω_n.
- And the third is the study of general, typical properties (valid for almost all) of the graphs under consideration. This approach essentially helps to understand the structure of graphs of the entire class, especially of the graphs with a large number of vertices.

Class of n-vertex graphs of a given diameter

The lecture discusses the indicated topic for the class of n-vertex graphs of a given diameter. Typical metric properties of these graphs related to

- variety of metric balls,
- graph radius,
- diametrical and central vertices,
- graph center and its spectrum (set of cardinalities of graph centers), etc., as well as
- classes of typical graphs arising here

are investigated.

Distance in a connected graph

Let us give definitions of the main metric characteristics of a graph. We study finite ordinary (without loops and multiple edges) graphs $G = (V, E)$ with vertex set $V = \{1, 2, \dots, n\}$, $n \in \mathbb{N}$. We use the generally accepted concepts and notations of graph theory [1,2], as well as the standard concepts of combinatorial analysis [3]. For a connected graph $G = (V, E)$ the distance $\rho_G(u, v)$ between its vertices $u, v \in V$ is defined as the length of the shortest path connecting these vertices.

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[1] V.A. Emelichev, O.I. Melnikov, V.I. Sarvanov, and R.I. Tyshkevich. *Lectures on Graph Theory* // B.I.Wissenschaftsverlag, Mannheim. 1994.
[2] F. Harary. *Graph Theory* // Addison–Wesley, Mass. 1969.
[3] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics* // Addison-Wesley. 1994.

In this case,

- $e_G(v) = \max_{u \in V} \rho_G(v, u)$ is the eccentricity of the vertex v of the graph G,
- $d(G) = \max_{v \in V} e_G(v)$ is the diameter of the graph G and
- $r(G) = \min_{v \in V} e_G(v)$ is the radius of the graph G.

A vertex is called central (diametral), if its eccentricity is equal to the radius (diameter), the shortest path of length $d(G)$ is called the diametral path of the graph G, and by a pair of diametrical vertices we mean an unordered sample of two vertices from the set V, the distance between which is equal to the diameter.

Extended metric

Let us extend the notion of the metric ρ_G (preserving all the metric axioms) to the case disconnected graphs in the standard way. We set $\rho_G(x, y) = \infty$ if a graph G does not have a path connecting the vertices $x, y \in V$.

For the extended metric, the diameter $d(G)$ and the radius $r(G)$ of a disconnected graph G will be equal to ∞.

We assume that $\infty + \infty = \infty$, $n + \infty = \infty$ and $\infty > n$ for every nonnegative integer n.

Concept of almost all graphs

In the asymptotic analysis of graphs, to estimate the measure of the number of graphs with a certain property, the concept of almost all is often used; in this approach, the studied property is considered for graphs with a large number of vertices.

Let $\mathcal{J}_n$ be the class of labeled n-vertex graphs with the fixed set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Consider some property $\mathcal{P}$, by which each graph may or may not possess. Through $\mathcal{J}_n^{\mathcal{P}}$ denote the set of all graphs from $\mathcal{J}_n$ that possess the property $\mathcal{P}$.

Almost all graphs possessing the property $\mathcal{P}$

- Almost all graphs possess the property $\mathcal{P}$ if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 1$, i.e. $|\mathcal{J}_n^{\mathcal{P}}| \sim |\mathcal{J}_n|$, and
- There are almost no graphs with the property $\mathcal{P}$ if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 0$.

In a similar way, we can talk about the property $\mathcal{P}$, which is possessed by almost all n-vertex graphs of the class Ω under study, or when there are almost no graphs in Ω that have property $\mathcal{P}$.

Here $\sim$ denotes *the asymptotic equality* as $n \to \infty$

Class of typical graphs of a given class

In the study and selection of almost all graphs in a class Ω of graphs under consideration it is often useful to define not characteristic properties themselves for the notion of almost all, but directly select a subclass of typical graphs itself.

Definition 1.

A subclass $\Omega^* \subseteq \Omega$ is the class of typical graphs of the class Ω if $|\Omega_n^*| \sim |\Omega_n|$, where $\Omega_n = \Omega \cap \mathcal{J}_n$ and $\Omega_n^* = \Omega^* \cap \mathcal{J}_n$.

Definition 2.

A property of graphs from a class under consideration is typical if almost all graphs from this class have the given property.

Let k be a positive fixed integer and $\mathcal{J}_{n,\,d=k}$, $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}^{*}_{n,\,d\geq k}$, $\mathcal{J}^{**}_{n,\,d\geq k}$ be the following classes of labeled n-vertex graphs:

- $\mathcal{J}_{n,\,d=k}$ is the class of n-vertex graphs of diameter k;
- $\mathcal{J}_{n,\,d\geq k}$ is the class of n-vertex connected graphs of diameter at least k;
- $\mathcal{J}^{*}_{n,\,d\geq k}$ is the class of n-vertex graphs (not necessarily connected) with the shortest path of length at least k;
- $\mathcal{J}^{**}_{n,\,d\geq k}$ is the class of n-vertex graphs (not necessarily connected) of diameter at least k.

Classes Inclusions

Obviously, the following inclusions are fulfilled:
$\mathcal{J}_{n,\,d=k} \subseteq \mathcal{J}_{n,\,d\geq k} \subseteq \mathcal{J}^{*}_{n,\,d\geq k} \subseteq \mathcal{J}^{**}_{n,\,d\geq k} \subseteq \mathcal{J}_{n}$.
Moreover, for $n > k+2$ all inclusions are strict (Figure 1).

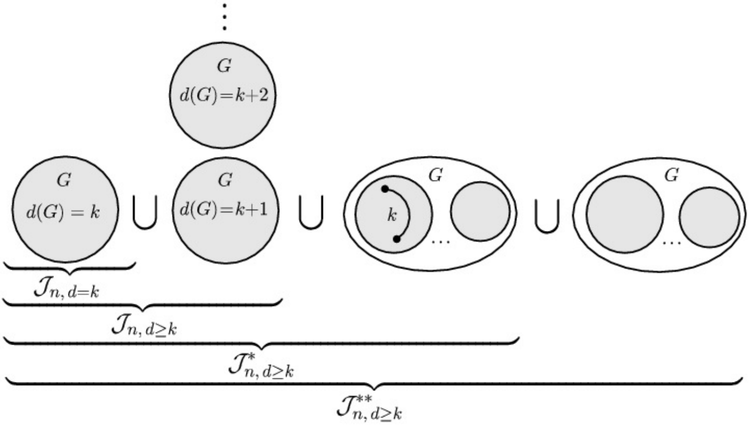

Fig. 1: Inclusions of classes

Therefore, when extending the class of graphs, the asymptotic order of the number of resulting graphs may change.

1. Number of graphs.

J.W. Moon and L. Moser established that almost all graphs are graphs of diameter 2 [4]. This means that for $k = 2$ all four classes of graphs $\mathcal{J}_{n,\,d=2}$, $\mathcal{J}_{n,\,d\geq 2}$, $\mathcal{J}^{*}_{n,\,d\geq 2}$, $\mathcal{J}^{**}_{n,\,d\geq 2}$ have the same asymptotic cardinality, the number of graphs in these classes is asymptotically equal to the number of all n-vertex graphs $2^{\binom{n}{2}}$.

Case $k = 2$ (J.W. Moon and L. Moser [4])

$$|\mathcal{J}_{n,\,d=2}| \sim |\mathcal{J}_{n,\,d\geq 2}| \sim |\mathcal{J}^{*}_{n,\,d\geq 2}| \sim |\mathcal{J}^{**}_{n,\,d\geq 2}| \sim |\mathcal{J}_{n}| = 2^{\binom{n}{2}}.$$

The asymptotics of the number of labeled n-vertex graphs of diameter $k \geq 3$ is found in [5].

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[4] J.W. Moon, L. Moser. *Almost all* $(0,1)$ *matrices are primitive* // Stud. Sci. Math. Hung. 1966. V.1. P. 153-156.
[5] Z. Füredi, Y. Kim. *The number of graphs of given diameter* // arXiv:1204.4580v1 [math.CO]. 2012.

Number of graphs

In [6,7] it is independently proved that for $k \geq 3$ all three classes of graphs $\mathcal{J}_{n,\, d=k}$, $\mathcal{J}_{n,\, d\geq k}$, $\mathcal{J}^*_{n,\, d\geq k}$ have the same asymptotic cardinality, and the asymptotically exact value $2^{\binom{n}{2}}\,\xi_{n,k}$ of the number of graphs from these classes is established, here

Case $k \geq 3$

$$\xi_{n,k} = q_k\,(n)_{k-1}\left(\frac{3}{2^{k-1}}\right)^{n-k+1}, \quad q_k = \frac{1}{2}\,(k-2)\,2^{-\binom{k-1}{2}},$$
$$(n)_k = n(n-1)\cdots(n-k+1),$$

and in this case we define $(n)_0 = (0)_0 = 1$ and $(n)_k = 0$ if $n < k$.

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[6] T.I. Fedoryaeva. *The diversity vector of balls of a typical graph of small diameter* // Diskretn. Anal. Issled. Oper. 2015. V.22(6). P.43–54.
[7] T.I. Fedoryaeva. *Asymptotic approximation for the number of n-vertex graphs of given diameter* // J.Appl.Ind.Math. 2017. V.11(2). P.204–214.

In the proof, the class $\mathcal{T}_{n,\,d=k}$ of typical n-vertex graphs of fixed diameter $k \geq 3$ was constructed (and hence, of classes $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}^*_{n,\,d\geq k}$) and we obtain the following asymptotically coinciding upper and lower estimates of the number of graphs of classes under consideration.

Upper and lower estimates of the number of graphs

Theorem 1 (number of graphs [6,7]).

Let $k \geq 3$ and $0 < \varepsilon < 1$ do not depend on n. Then there exists a constant $c > 0$ independent of n and such that for every $n \in \mathbb{N}$ the following inequalities hold

$$2^{\binom{n}{2}}\xi_{n,k}\left(1 - c\left(\frac{5+\varepsilon}{6}\right)^{n-k+1}\right) \leq |\mathcal{T}_{n,\, d=k}| \leq |\mathcal{J}_{n,\, d=k}| \leq |\mathcal{J}_{n,\, d\geq k}| \leq |\mathcal{J}^*_{n,\, d\geq k}| \leq 2^{\binom{n}{2}}\xi_{n,k}\left(1 + c\left(\frac{5+\varepsilon}{6}\right)^{n-k+1}\right).$$

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[6] T.I. Fedoryaeva. *The diversity vector of balls of a typical graph of small diameter* // Diskretn. Anal. Issled. Oper. 2015. V.22(6). P.43–54.
[7] T.I. Fedoryaeva. *Asymptotic approximation for the number of n-vertex graphs of given diameter* // J.Appl.Ind.Math. 2017. V.11(2). P.204–214.

Upper estimate of the number of graphs for $k = 3$

Note that for $k = 3$ the upper bound in Theorem 1 takes the form $2^{\binom{n}{2}}\xi_{n,3}$. Moreover, this upper estimate is valid even for the class of graphs containing all disconnected graphs (which do not necessarily have a connected component with the shortest path of length 3), namely, for the superclass $\mathcal{J}^{**}_{n,\,d\geq 3}$ of the class $\mathcal{J}^{*}_{n,\,d\geq 3}$.

Case $k = 3$

Corollary 1 (case $k = 3$ [6]).

Let $0 < \varepsilon < 1$ be independent of n. Then there is a constant $c > 0$ independent of n and such that for every $n \in \mathbb{N}$ the following inequalities hold

$$2^{\binom{n}{2}}\xi_{n,3}\left(1 - c\left(\frac{5+\varepsilon}{6}\right)^{n-2}\right) \leq |\mathcal{T}_{n,\,d=3}| \leq |\mathcal{J}_{n,\,d=3}| \leq |\mathcal{J}_{n,\,d\geq 3}| \leq |\mathcal{J}^{*}_{n,\,d\geq 3}| \leq |\mathcal{J}^{**}_{n,\,d\geq 3}| \leq 2^{\binom{n}{2}}\xi_{n,3}.$$

Thus, when extending the class of connected graphs $\mathcal{J}_{n,\,d\geq k}$ to the class $\mathcal{J}^{**}_{n,\,d\geq k}$ (by adding all disconnected graphs) for $k=3$ we also get the asymptotically equivalent class [6], and for $k\geq 4$ the class of higher asymptotic order arises [7].

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[6] T.I. Fedoryaeva. *The diversity vector of balls of a typical graph of small diameter* // Diskretn. Anal. Issled. Oper. 2015. V.22(6). P.43–54.
[7] T.I. Fedoryaeva. *Asymptotic approximation for the number of n-vertex graphs of given diameter* // J.Appl.Ind.Math. 2017. V.11(2). P.204–214.

2. Diversity of metric balls.

Recall that by the ball (sphere) of the graph $G = (V, E)$ we mean the metric ball (sphere) in the metric space of the graph G with the path metric $\rho_G : V \to \mathbb{Z}_{\geq 0}$, i.e.

- $B_i^G(v) = \{u \in V \mid \rho_G(v, u) \leq i\}$ is a ball of radius i centered at vertex $v \in V$ and
- $S_i^G(v) = \{u \in V \mid \rho_G(v, u) = i\}$ is a sphere of radius i centered at v.

Let $i \geq 0$ and $\tau_i(G)$ be the number of different balls of radius i contained in the graph G.

Definition 3.

For a connected graph G of diameter k, the numbers $\tau_i(G)$, $i \geq 0$ form a vector $\tau(G) = (\tau_0(G), \tau_1(G), \ldots, \tau_k(G))$, which we call the diversity vector of balls of the graph G [8].

Definition 4.

A graph G is said to have local t-diversity of balls if $\tau_0(G) = \tau_1(G) = \ldots = \tau_t(G)$. A graph with a local $(k-1)$-variety of balls is called a graph of the full diversity of balls.

Note that $\tau_0(G) = |V|$ and $\tau_i(G) = 1$ for $i \geq k$, and the diversity vector of balls of an n-vertex graph of diameter k with full diversity of balls has the form $(n, n, \ldots, n, 1)$.

[8] T.I. Fedoryaeva. *The diversity of balls in metric spaces of trees* // Diskretn. Anal. Issled. Oper. 2005. V.12(3). P.74–84.

Calculation of the diversity vector of balls of a given graph (not necessarily connected)

In [9], the notion of the diversity vector of balls was naturally extended to the case of disconnected graphs. There also a criterion of the coincidence of balls with centers at different vertices was proved and we developed an algorithm of calculating the diversity vector of balls of a given graph $G = (V, E)$ with a running time of $O(|V|^3)$.

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[9] T.I. Fedoryaeva. *Calculation of the diversity vector of balls of a given graph* // Siber. Electr. Math. Reports. 2016. V.13. P.122–129.

Various properties of ball diversity vectors and their components have been established both for all graphs and for separate classes. Let us consider only some extreme cases.

For n-vertex graphs of diameter k with local t-diversity of balls, a lower bound for the number of vertices is valid, which is defined in terms of the parameters k and t [10], the extremal graphs arising here are described up to isomorphism in [11].

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[10] T.I. Fedoryaeva. *Diversity vectors of balls in graphs and estimates of the components of the vectors* // J.Appl.Ind.Math. 2008. V.2(3). P.341–356.
[11] T.I. Fedoryaeva. *On graphs with given diameter, number of vertices, and local diversity of balls* // J.Appl.Ind.Math. 2011. V.5(1). P.44–50.

Exact upper estimates and exact lower estimates

In [10, 12], based on the investigation of the location of centers of different balls, the author obtained exact upper estimates $\overline{\tau}_i\left(\mathcal{J}_{n,\,d=k}\right)$ and exact lower estimates $\underline{\tau}_i\left(\mathcal{J}_{n,\,d=k}\right)$ of the number of different balls of given radius i in n-vertex graphs of diameter k. Wherein majorants and minorants (extremal graphs on which all upper and all lower bounds are achieved, respectively) were studied in [10, 13]. Similar results were obtained for the classes of connected n-vertex trees, all n-vertex connected graphs, and etc. (i.e. in graphs of fixed volume) [10].

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[10] T.I. Fedoryaeva. *Diversity vectors of balls in graphs and estimates of the components of the vectors* // J.Appl.Ind.Math. 2008. V.2(3). P.341–356.

[12] Т.И. Федоряева. *Точные верхние оценки числа различных шаров заданного радиуса в графах с фиксированными числом вершин и диаметром* // Дискретн. анализ и исслед. опер. 2009. Т.16(6). С.86–104.

[13] T.I. Fedoryaeva. *Majorants and minorants for the classes of graphs with fixed diameter and number of vertices* // J.Appl.Ind.Math. 2013. V.7(2). P.153–165.

Theorem 2 [10,12].

For every n-vertex graph G of diameter k, the following inequalities hold

$$\underline{\tau}_i\left(\mathcal{J}_{n,\, d=k}\right) \leq \tau_i(G) \leq \overline{\tau}_i\left(\mathcal{J}_{n,\, d=k}\right),$$

where

$$\underline{\tau}_i\left(\mathcal{J}_{n,\, d=k}\right) = \begin{cases} n, & \text{if } i = 0, \\ \Delta_i^k, & \text{if } 1 \leq i < k, \\ 1, & \text{if } i \geq k; \end{cases}$$

$$\Delta_i^k = \begin{cases} k+1, & \text{if } 0 \leq i \leq \lfloor k/2 \rfloor, \\ 2(k-i)+1, & \text{if } \lfloor k/2 \rfloor < i < k, \\ 1, & \text{if } i \geq k; \end{cases}$$

here $\Delta_k = (\Delta_0^k, \Delta_1^k, \ldots, \Delta_k^k)$ *is the diversity vector of balls of a simple path of length k;*

Exact upper estimates and exact lower estimates

Theorem 2.

$$\overline{\tau}_i\left(\mathcal{J}_{n,\,d=k}\right) = \begin{cases} n, & \textit{if } 0 \le i < k \\ & \quad \textit{and } i \le \max\{\lfloor k/2 \rfloor, s\}, \\ 3(k-i)+1, & \textit{if } \lfloor k/2 \rfloor < s < i < k, \\ n+k+\lfloor k/2 \rfloor - 3i, & \textit{if } s \le \lfloor k/2 \rfloor < i \le \lfloor k/2 \rfloor + s \\ & \quad \textit{and } i < k, \\ 2(k-i)+1, & \textit{if } \lfloor k/2 \rfloor + s < i < k, \\ 1, & \textit{if } i \ge k, \end{cases}$$

$s = n - k - 1.$

Class $\mathcal{F}_{n,k}$

In an asymptotic study of the diversity of balls, the author distinguished a new class $\mathcal{F}_{n,k}$ of typical n-vertex graphs of diameter $k \geq 3$, which is a subclass of the previously constructed class $\mathcal{T}_{n,\, d=k}$ and has a number of metric properties [14]. It turned out that a number of typical properties connected with the diversity of metric balls contained in the graph hold.

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[14] T.I. Fedoryaeva. *Structure of the diversity vector of balls of a typical graph with given diameter* // Siber. Electr. Math. Reports. 2016. V.13. P.375–387.

Typical diversity of metric balls

So, for every $k \geq 1$, a set of integer vectors $\Lambda_{n,k}$ of length $k+1$ (consisting of $\lfloor \frac{k-1}{2} \rfloor$ different vectors for $k \geq 5$ and a unique vector for $k < 5$), which establishes the typical diversity of metric balls of n-vertex graphs of diameter k.

In addition, the location of the centers of all different balls of given radius for almost all such graphs is established. In the proof of this fact, the previously mentioned criterion of the coincidence of balls is used [9].

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[9] T.I. Fedoryaeva. *Calculation of the diversity vector of balls of a given graph* // Siber. Electr. Math. Reports. 2016. V.13. P.122–129.

Theorem 3 (typical diversity of balls [14]).

The diversity vector of balls of almost all n-vertex graphs of fixed diameter k belongs to $\Lambda_{n,k}$.

For example, for $k = 1, 2, 3, 4$, the set $\Lambda_{n,k}$ contains the only unique vector:

- $\Lambda^0_{n,1} = (n, 1)$ if $k = 1$,
- $\Lambda^0_{n,2} = (n, n, 1)$ if $k = 2$,
- $\Lambda^0_{n,3} = (n, n, 3, 1)$ if $k = 3$ and
- $\Lambda^0_{n,4} = (n, n, 5, 3, 1)$ if $k = 4$.

The case $k \geq 5$ already leads to a more complicated description of the vectors of the set $\Lambda_{n,k}$.

The following typical properties of t-diversity of balls follow directly from Theorem 3.

Corollary 2.

For $k \geq 3$ almost all n-vertex graphs of diameter k do not have the local 2-diversity of balls (and, in particular, do not have the full diversity of balls), wherein have the local 1-diversity of balls.

Corollary 3.

Almost all n-vertex graphs of fixed diameter $k = 1, 2$ have the full diversity of balls.

Thus, the set of vectors $\Lambda_{n,k}$ gives all diversity vectors of balls of a typical graph of fixed diameter k. We also note that this property is violated when any vector is removed from the set $\Lambda_{n,k}$ (see Theorem 6 in [14]), i.e., in this sense, Theorem 3 cannot be improved.

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[14] T.I. Fedoryaeva. *Structure of the diversity vector of balls of a typical graph with given diameter* // Siber. Electr. Math. Reports. 2016. V.13. P.375–387.

3. Number of pairs of diametral vertices.

When studying typical graphs and their properties for the class $\mathcal{J}_{n,\,d=k}$, it turned out that for $k \geq 3$ almost all n-vertex graphs of diameter k have a unique pair of diametrical vertices.

However, for graphs of diameter 2, it was found that this property does not hold, i.e., the number of n-vertex graphs of diameter 2 with a unique pair of diametral vertices is asymptotically small.

This prompted a detailed study of more general properties of graphs of diameter 2, related to the number of pairs of diametrical vertices, contained in the graph. In particular, one of the questions that arises here is whether it is possible to limit the number of pairs of diametral vertices to receive almost all graphs of diameter 2.

Recall that by a pair of diametrical vertices we mean an unordered sample of two vertices, the distance between which is equal to the diameter.

In addition, the class $\mathcal{J}_{n,\,d=2}$ has always been of particular interest in view of, on the one hand, the apparent simplicity of its objects, on the other hand, the breadth of its "coverage" of all graphs: recall that almost all graphs have a diameter 2 (J.W. Moon, L. Moser [4]).

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[4] J.W. Moon, L. Moser, *Almost all* $(0,1)$ *matrices are primitive*, Stud. Sci. Math. Hung., **1** (1966), 153-156.

Classification of graphs of diameter 2

In this connection, a more subtle classification of graphs of diameter 2 is interesting, when

- subclasses are distinguished that form a partition of the entire class $\mathcal{J}_{n,\, d=2}$.
- Moreover, for meaningful classification it is required that none of the considered subclasses would not be poor and too rich, i.e. asymptotically did not coincide with the whole class.

At the same time, considering the "wealth" of the whole class $\mathcal{J}_{n,\, d=2}$, apparently we should not expect a good characterization of the selected subclasses.

However, there is a natural problem of describing or constructing a class of typical graphs inside each studied subclass in order to clarify the structure of such graphs with a large number of vertices.

Classification of graphs of diameter 2

Thus, we have the following tasks

- classification of graphs of diameter 2 and
- constructing a class of typical graphs for each distinguished subclass of graphs of diameter 2.

In [15] the classification of graphs of diameter 2 by the number of pairs of diametral vertices contained in the graph is designed by the author.

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[15] T.I. Fedoryaeva. *Classification of graphs of diameter* 2 // Siber. Electr. Math. Reports. 2020. V.17. P.502–512.

Definition 5 (class $\mathcal{J}_{n,\,q=k}$).

For any nonnegative integer k by $\mathcal{J}_{n,\,q=k}$ we denote the class of all n-vertex labeled graphs of diameter 2 containing exactly k pairs of diametral vertices.

Partition of class $\mathcal{J}_{n,\,d=2}$ of graphs of diameter 2

It's obvious that

$$\mathcal{J}_{n,\,d=2} = \bigcup_{k\geq 0} \mathcal{J}_{n,\,q=k}.$$

Moreover, $\mathcal{J}_{n,\,q=0} = \varnothing$ and nonempty subclasses $\mathcal{J}_{n,\,q=k}$ form a partition of the class of all n-vertex graphs of diameter 2.

Firstly, we find out when in the class of n-vertex graphs of diameter 2 there exists a graph containing exactly k pairs of diametral vertices.

Example: class $\mathcal{J}_{n,\,q=1}$

For k = 1 graphs of class $\mathcal{J}_{n,\,q=k}$ are arranged quite simply:

Case $k = 1$

(i) For $n \geq 3$, $\mathcal{J}_{n,\,q=1}$ consists of graphs isomorphic $\overline{K}_2 + K_{n-2}$;
(ii) $\mathcal{J}_{n,\,q=1} = \varnothing$ if $n < 3$.

The generally accepted notations of graph theory are used here:

- $\overline{G}$ — complement of the graph G,
- $G + H$ — the graph obtained by the join operation from graphs G and H,
- K_n — complete n-vertex graph.

The general case is solved in the next theorem, where the necessary and sufficient condition under which $\mathcal{J}_{n,\,q=k} \neq \varnothing$ is established.

Theorem 4 (condition of $\mathcal{J}_{n,\,q=k} \neq \varnothing$ [15]).

Let $k \geq 1$. Then there exists a graph containing exactly k pairs of diametral vertices in the class of n-vertex graphs of diameter 2 iff $n \geq \lceil 0.5(3 + \sqrt{1+8k}) \rceil$.

Number of graphs of class $\mathcal{J}_{n,\,q=k}$

The previous theorem gives the smallest order of graphs of the class $\mathcal{J}_{n,\,q=k}$. Further we also investigate such graphs with a large number of vertices; they are counted in the following theorem.

Theorem 5 (number of graphs of class $\mathcal{J}_{n,\,q=k}$ [15]).

Let $k \geq 1$ be a given integer. Then the number of labeled n-vertex graphs of diameter 2 containing exactly k pairs diametral vertices is equal to $\binom{\binom{n}{2}}{k}$ for every $n > k + 1$.

Note that the estimate $n > k + 1$ in Theorem 5 is unimprovable in the sense of the above equality $|\mathcal{J}_{n,\,q=k}| = \binom{\binom{n}{2}}{k}$.

Typical graphs of class $\mathcal{J}_{n,\,q=k}$

After the partition of the graphs class of diameter 2 into subclasses and counting the number of graphs with a large number of vertices in them, we pass to the problem of constructing a class of typical graphs for each defined subclass of graphs of diameter 2. Distinguished typical graphs of class $\mathcal{J}_{n,\,q=k}$ are given in the following theorem.

Theorem 6 (class $\mathcal{J}^{\top}_{n,\,q=k}$ of typical graphs [15]).

Let $k \geq 1$ be any fixed integer. Then graphs isomorphic $\overline{kK}_2 + K_{n-2k}$ for $n \neq 2$ form a class of typical graphs of class $\mathcal{J}_{n,\,q=k}$.

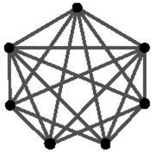
After constructing the subclass $\mathcal{J}^{\top}_{n,\,q=k}$ of typical graphs for the class $\mathcal{J}_{n,\,q=k}$ naturally we want to understand what is left in the class of graphs outside of typical graphs, i.e. how much we "deepened" into the class itself.

Example (class $\mathcal{J}^{\perp}_{n,\,q=k}$ of atypical graphs)

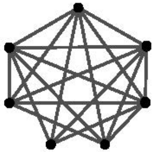
For any fixed $k \geq 2$ in class $\mathcal{J}_{n,\,q=k}$ there is almost no graphs are isomorphic $\overline{K}_{1,k} + K_{n-(k+1)}$.

Note also that any subclass of the class $\mathcal{J}_{n,\,q=k}$ that has an asymptotically "small" intersection with the constructed class of typical graphs possesses a similar atypical property.

The following figure shows examples of typical graph $\overline{2K}_2 + K_3 \in \mathcal{J}^{\top}_{7,\,q=2}$ and atypical graph $\overline{K}_{1,2} + K_4 \in \mathcal{J}^{\perp}_{7,\,q=2}$ for class $\mathcal{J}_{7,\,q=2}$.



$\overline{2K_2} + K_3 \in \mathcal{J}^{\top}_{7,q=2}$



$\overline{K}_{1,2} + K_4 \in \mathcal{J}^{\perp}_{7,q=2}$

Fig. 2: Graphs in class $\mathcal{J}_{7,\,q=2}$

Now let's turn to the question of the possibility of limiting the number of pairs of diametral vertices sufficient to obtain almost all graphs diameter 2. And consider the more general case when $q = q(n)$ is a function depending on n and taking nonnegative integer values with the growth restriction under consideration, covering the case of fixed integer $q \geq 1$.

Theorem 7 (case $q = q(n)$ [15]).

Let $0 < q(n) \leq \varepsilon\binom{n}{2}$ as $n \to \infty$, where ε is a fixed constant and $0 < \varepsilon < \frac{1}{2}$. Then almost all n-vertex graphs of diameter 2 have at least $q(n)$ pairs of diametral vertices.

In particular, the Theorem 7 means that it is impossible to limit the number of pairs of diametral vertices by a given fixed integer in order to obtain almost all graphs of diameter 2.

Corollary 4.

For any fixed integer $q \geq 1$, almost all graphs of diameter 2 have at least q pairs of diametral vertices.

Relation between radius and diameter

4. Radius of a graph.

The relation

$$r(G) \leq d(G) \leq 2r(G)$$

between the radius and the diameter of an arbitrary connected graph G is well known.

Moreover, for every r and d satisfying the relation $r \leq d \leq 2r$ and every $n \geq d + r$, F. Ostrand's theorem implies the existence of an n-vertex graph G with $r(G) = r$ and $d(G) = d$ [16].

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[16] P.A. Ostrand. *Graphs with specified radius and diameter* // Discrete Math. 1973. V.4. P.71-75.

From the result of J.W. Moon and L. Moser, it is easy to obtain that almost all n-vertex graphs have diameter and radius equal to 2 [1].

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[1] V.A. Emelichev, O.I. Melnikov, V.I. Sarvanov, and R.I. Tyshkevich. *Lectures on Graph Theory* // B.I.Wissenschaftsverlag, Mannheim. 1994.

Radius and diameter of a random graph

Yu.D. Burtin considering a random n-vertex graph $G_p(n)$ with the probability $p = p(n)$ of the presence of an edge, showed that

$$L_p(n) + 1 \leq r(G_p(n)) \leq d(G_p(n)) \leq L_p(n) + 2$$

under natural restrictions on the growth of $p(n)$ if $n \to \infty$, i.e. radius and diameter of the random graph $G_p(n)$ can take only two values $L_p(n)+1$ and $L_p(n)+2$, which are calculated in [17], with probability tending to 1 as $n \to \infty$.

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[17] Yu.D. Burtin. *On Extreme Metric Parameters of a Random Graph //* Asymptotic Estimates, Theory Probab. Appl. 1975. T.19(4). C.710-725.

The question naturally arises about a possible radius of almost all n-vertex graphs of diameter k.

Note that for the probability space of n-vertex graphs G of a class Ω under consideration (for almost all graphs) in the distribution of values of a studied numerical graph characteristic

$$\mathcal{X} : \Omega_n \to \mathbb{R},$$

generally speaking, a unique value of this characteristic $\mathcal{X}(G)$ does not necessarily arise (this situation is known for a number of characteristics of a graph, further we will see such an effect in the study of graph center).

Radius of almost all n-vertex graphs of diameter $k = 1$

From the relationship between radius $r(G)$ and diameter k it follows that the radius can take on a value between

$$\left\lceil \frac{k}{2} \right\rceil \leq r(G) \leq k.$$

Note that complete graph K_n is the unique n-vertex graph of diameter 1 and $r(K_n) = 1$. Therefore, almost all graphs of diameter $k = 1$ have the radius equal to the diameter.

A similar fact for graphs of diameter 2 also trivially follows from the well-known above theorems.

Theorem 8 (case $k = 2$).

Almost all n-vertex graphs of diameter $k = 2$ have radius 2.

Class $\mathcal{F}_{n,k,p}$

When studying the radius of graphs of diameter $k \geq 3$, in the class $\mathcal{F}_{n,k}$ we distinguish a family $\mathcal{F}_{n,k,p}$, $p \geq 1$ of nested subclasses of n–vertex graphs [18]:

$$\mathcal{F}_{n,k} \supseteq \mathcal{F}_{n,k,1} \supseteq \ldots \supseteq \mathcal{F}_{n,k,p} \supseteq \mathcal{F}_{n,k,p+1} \supseteq \ldots.$$

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[18] T.I. Fedoryaeva. *On radius and typical properties of n-vertex graphs of given diameter* // Siber. Electr. Math. Reports. 2021. V.18. P.345–357.

Each of these classes preserves the already established properties of graphs from $\mathcal{F}_{n,k}$; in addition, the graphs of the introduced classes have a property of metric spheres, which ensures the presence of a predetermined number of vertices in the intersection of spheres of radius 1.

Asymptotically exact value of $|\mathcal{F}_{n,k,p}|$

Theorem 9 (класс $\mathcal{F}_{n,k,p}$ [18].)

Let $k \geq 3$, $0 < \varepsilon < 1$ and $p \geq 1$ do not depend on n. Then there exists a constant $c > 0$ independent of n and such that for every $n \in \mathbb{N}$ the following inequalities hold

$$2^{\binom{n}{2}} \xi_{n,k} \left(1 - c\left(\frac{5+\varepsilon}{6}\right)^{n-k+1}\right) \leq |\mathcal{F}_{n,k,p}|$$

$$\leq |\mathcal{F}_{n,k}| \leq |\mathcal{T}_{n,\, d=k}| \leq |\mathcal{J}_{n,\, d=k}| \leq |\mathcal{J}_{n,\, d\geq k}|$$

$$\leq |\mathcal{J}^*_{n,\, d\geq k}| \leq 2^{\binom{n}{2}} \xi_{n,k} \left(1 + c\left(\frac{5+\varepsilon}{6}\right)^{n-k+1}\right).$$

Corollary 5.

Let $k \geq 3$ and $p \geq 1$ be independent of n. Then the following asymptotic equalities hold as $n \to \infty$

$$|\mathcal{F}_{n,k,p}| \sim |\mathcal{F}_{n,k}| \sim |\mathcal{J}_{n,\, d=k}| \sim |\mathcal{J}_{n,\, d\geq k}| \sim |\mathcal{J}^*_{n,\, d\geq k}| \sim 2^{\binom{n}{2}} \xi_{n,k}.$$

What does class $\mathcal{F}_{n,k,p}$ give us?

Thus, for any fixed $p \geq 1$, the class of graphs $\mathcal{F}_{n,k,p}$ is the class of typical graphs of each of the classes $\mathcal{J}_{n,\,d=k}$, $\mathcal{J}_{n,\,d\geq k}$ and $\mathcal{J}^*_{n,\,d\geq k}$.

The introduced condition on the spheres of a graph of the class $\mathcal{F}_{n,k,p}$ ensures the existence of vertices of a large degree and a wide variety of short shortest paths, provided, in a certain sense, of the "uniqueness" (up to a segment of two vertices) of the longest shortest path for almost all graphs of diameter k. This condition also turns out to be useful in studying typical properties of n-vertex graphs associated with various metric characteristics. Note that these classes $\mathcal{F}_{n,k,p}$ will also be used further when studying the graph center.

Radius of almost all n-vertex graphs of diameter $k \geq 3$

Based on the found typical properties of metric balls contained in the graph and Theorem 9, a typical radius is found.

Theorem 10 (typical radius [18]).

The radius of almost all n-vertex graphs of fixed diameter $k \geq 3$ *is equal to* $\lceil \frac{k}{2} \rceil$.

Thus, the radius of almost all graphs of a given fixed diameter is established in Theorems 8 and 10.

[18] T.I. Fedoryaeva. *On radius and typical properties of n-vertex graphs of given diameter* // Siber. Electr. Math. Reports. 2021. V.18. P.345–357.

Recall that a graph is called self-centered [19] if all its vertices are central.

Corollary 6.

There are almost no self-centered n-vertex graphs of fixed diameter $k \geq 3$*, while almost all graphs of diameter* $k = 1, 2$ *are self-centered.*

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[19] F. Buckley, F. Harary. *Distance In Graphs* // Addison-Wesley Publishing Company, Redwood City. 1990.

All obtained typical properties for n-vertex graphs of fixed diameter $k \geq 2$ remain typical for connected graphs of diameter at least k, as well as for graphs (not necessarily connected) containing a shortest path of a length at least k. In particular, the following corollary holds.

Corollary 7.

For every fixed $k \geq 3$, almost all n-vertex graphs of each of the following classes $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}^*_{n,\,d\geq k}$ are connected and have radius $\lceil \frac{k}{2} \rceil$, diameter k.

5. Center of the graph.

Definition 6.

The set of central vertices of a graph G forms its center $\mathbb{C}(G)$.

The concept of the graph center is related to its numerous practical problems arising in varied fields and it is stipulated by the measurement of proximity centrality in the analysis of various kinds of networks and connections. Often in graphs corresponding to communication networks, the diameter is interpreted as the time of data transfer or the path length between the most distant nodes, the radius is as the reachability time from the most distant nodes to the central node, serving as the main distribution center.

In this case, the presence of several such centers is allowed (see, for example, [20]). Finding of the graph center turns out to be useful for problems of optimal placement of publicly important institutions and enterprises (hospitals, fire stations, post offices and other emergency points), when it is required to minimize the farthest distances to these institutions [19]. So, the location of the hospital at the central vertices of the graph that arises here reduces the maximum distance that ambulances have to cover.

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[19] F. Buckley, F. Harary. *Distance In Graphs* // Addison-Wesley Publishing Company, Redwood City. 1990.
[20] A.M. Rappoport. *Metric characteristics in the communication networks* // Proceedings of ISA RAN. 2005. V.14. P.141-147.

The concept of centrality

The concept of centrality is also applied in social sciences and is actively used in the analysis of social networks [21], for example, when the most influential persons of considered network are identified. In biology, it is relevant when building models of the spread of diseases, in chemistry when analyzing molecular bonds, etc.

[21] S. Wasserman, K. Faust. *Social network analysis: methods and applications* // Cambridge University Press, Cambridge. 1994.

A few results about the graph center

A number of classical results about the graph center are well known [1, 22–24]. So, the realizability of an arbitrary graph as a subgraph generated by the center of a suitable graph is established. Namely, it is proved that for any graph H there exists a connected graph G such that its subgraph generated by the center $\mathbb{C}(G)$ is isomorphic to H. This fact was established by G.N. Kopylov and E.A. Timofeev [22], its simple justification was also given by S.T. Hedetniemi (see [23]).

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[1] V.A. Emelichev, O.I. Melnikov, V.I. Sarvanov, and R.I. Tyshkevich. *Lectures on Graph Theory* // B.I.Wissenschaftsverlag, Mannheim. 1994.
[22] G.N. Kopylov, E.A. Timofeev. *Centers and radii of graphs* // Uspekhi Mat. Nauk. 1977. V.32(6). P.226.
[23] F. Buckley, Z. Miller, P.J. Slater. *On graphs containing a given graph as a center* // J. Graph Theory, 1981. V.5. P. 427-434.
[24] F. Buckley. *The central ratio of a graph* // Discrete Math. 1982. V.38(1). P.17-21.

Central ratio

F. Buckley investigated the so-called central ratio

$$\mathbb{R}_c(G) = \frac{|\mathbb{C}(G)|}{|V(G)|}$$

of connected graph G, for which it is obvious that the inequality

$$0 < \mathbb{R}_c(G) \leq 1$$

holds and $\mathbb{R}_c(G) \in \mathbb{Q}$. For any rational q in the interval $(0, 1]$, he proved the existence of a graph G such that $\mathbb{R}_c(G) = q$ [24].

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[24] F. Buckley. *The central ratio of a graph* // Discrete Math. 1982. V.38(1). P.17-21.

Structure of the center

In [25], the structure of the center of almost all n-vertex graphs of diameter k was studied. For such graphs for $k = 1, 2$ any vertex is central, i.e. $\mathbb{C}(G) = V(G)$. But for $k \geq 3$ we identified two types of central vertices, which are necessary and sufficient to obtain the centers of almost all n-vertex graphs of fixed diameter k.

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[25] T.I. Fedoryaeva. *Center and its spectrum of almost all n-vertex graphs of given diameter* // Siber. Electr. Math. Reports. 2021. V.18. P.511–529.

Typical center

Theorem 11 (typical center [25].)

Let $k \geq 3$ be a fixed integer. Then
(i) *the center of almost all n-vertex graphs of even diameter k consists of all central vertices of diametral paths of the graph*;
(ii) *the center of almost all n-vertex graphs of odd diameter k consists of all central vertices of diametral paths of the graph and all vertices equidistant at distance $\frac{k+1}{2}$ from their endpoints. Furthermore, the proportion of such n-vertex graphs whose center consists only of central vertices of diametral paths of the graph is asymptotically equal to $\frac{k-3}{k-2}$.*

Typical center

Note that in the center $\mathbb{C}(G)$ one cannot do without vertices of each of the distinguished types of central vertices in order to obtain almost all graphs G of diameter $k \geq 3$. This follows directly from Theorem 11 for $k \geq 4$, and additionally for $k = 3$ the theorem of F. Buckley and M. Levinter on L'-graphs [26] is used.

Note also that the center of typical graphs from the class $\mathcal{F}_{n,k,p}$ is explicitly distinguished.

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[26] F. Buckley, M. Lewinter. *Graphs with all diametral paths through distant central nodes*//Mathematical and Computer Modelling. 1993. V.17(11).P.35-41.

6. Center spectrum.

Definition 7 (center spectrum [25]).

For an arbitrary class of connected graphs Ω through $\mathbb{S}p_c(\Omega)$ we denote the center spectrum of graphs of this class, i.e. the set of cardinalities of graphs centers from class Ω.

The class of all n-vertex connected graphs naturally partitioned into subclasses of graphs determined by their diameter. It is obvious that

- $\mathcal{J}_{n=1,\, d=k=0} = \{K_1\}$ and
- $\mathcal{J}_{n,\, d=k=0} = \varnothing$ if $n \neq 1$, and
- $\mathcal{J}_{n,\, d\geq 1} = \bigcup_{k\geq 1} \mathcal{J}_{n,\, d=k}$ is the class of all nontrivial connected graphs.

Theorem of G.N. Kopylov and E.A. Timofeev

In [22], G.N. Kopylov and E.A. Timofeev found all possible values of the parameters n, m, and c for which there exists an n-vertex graph with m edges and c central vertices. The relations between these parameters are reduced to lower and upper estimates of the number of edges m in terms of the given parameters n and c.

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[22] Г.Н. Копылов, Е.А. Тимофеев. *О центрах и радиусах графов* // УМН. 1975. Т.32(6). С.226.

Theorem of G.N. Kopylov and E.A. Timofeev

Theorem 12 [22].

For $n \geq 2$, there exists a graph in the class of connected n-vertex graphs with m edges and c central vertices if and only if, when one of the following conditions holds:

(i) $c \leq n$, $c \neq n-1$ *and* $\frac{c(c-1)}{2} + c(n-c) \leq m \leq \frac{n(n-2)+c}{2}$;

(ii) $c = n$ *and* $n \leq m \leq \frac{n(n-2)}{2}$;

(iii) $2 \leq c \leq n-2$ *and* $E(c,n) \leq m \leq \frac{(n-2)(n-3)}{2} + c$, *where*

$$E(c,n) = \begin{cases} n-1, & \text{if } c = 2, \\ n+1, & \text{if } c \text{ even and } n-3 \leq c \leq n-2, \\ n, & \text{else.} \end{cases}$$

Using this Theorem, one can find the center spectrum of all n-vertex connected graphs. The following theorem describes the spectrum of the center of all and almost all graphs of the class $\mathcal{J}_{n,\, d\geq 1}$.

Theorem 13 (spectrum $\mathbb{S}p_c(\mathcal{J}_{n,\, d\geq 1})$ [25]).

The following properties are valid:
(i) $\mathbb{S}p_c(\mathcal{J}_{n,\, d\geq 1}) = [\,[\,1,\, n\,]\,] \setminus \{n-1\}$ *for every* $n \geq 2$;
(ii) *almost all* n*-vertex connected graphs have the center of cardinality* n.

Here $[\,[\,x,\, y\,]\,]$ denotes an integer interval between two given numbers $x, y \in \mathbb{R}$, i.e. $[\,[\,x,\, y\,]\,] = [x,\, y] \cap \mathbb{Z}$.

Center Spectrum of almost all n-vertex graphs of diameter k?

- Thus, for almost all n-vertex connected graphs G, the following equality holds $|\mathbb{C}(G)| = n$.
- It's also obvious that $\mathbb{S}p_c(\mathcal{J}_{n,\, d=1}) = \{n\}$ if $n \geq 2$.
- In addition, Corollary 6 implies that the cardinality of the center of almost all graphs in $\mathcal{J}_{n,\, d=2}$ is also equal to n.

The question naturally arises about a possible center spectrum of almost all n-vertex graphs of diameter $k \geq 3$.

Note that Yanan Hu and Xingzhi Zhan found the center spectrum of n-vertex graphs of given radius r [27]. As for the properties of the center spectrum of almost all n-vertex graphs of fixed diameter k (for large n), this result only implies inclusion

$$\mathbb{S}p_c(\mathcal{J}_{n,\, d=k}) \subseteq [[\, 1,\ n \,]] \setminus \{n-1\} \text{ for all } n \geq (8k-2)/3.$$

This relation is a consequence of equality in Theorem 13(i), i.e. new restrictions for possible values of the center spectrum of the almost all graphs do not arise.

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[27] Yanan Hu, Xingzhi Zhan. *Possible cardinalities of the center of a graph* // arXiv:2009.05925 [math.CO] (2020).

The smallest and the largest number of vertices with eccentricity c in n-vertex graphs of diameter k

In [28], Dhruv Mubayi and Douglas B. West investigated the smallest $h_{n,k}(c)$ and the largest $f_{n,k}(c)$ number of vertices with eccentricity c in n-vertex graphs of diameter k. For individual cases that do not cover all possible relationships between parameters n, k and c, the values $f_{n,k}(c)$ and $h_{n,k}(c)$ are found. In particular, for $c = \lceil \frac{k}{2} \rceil$, which is the radius of almost all graphs from $\mathcal{J}_{n,\, d=k}$, we have

$$h_{n,k}(\lceil \frac{k}{2} \rceil) = 0,$$

$$f_{n,k}(\frac{k}{2}) = n - k,\ f_{n,k}(\frac{k+1}{2}) = n - k + 1.$$

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[28] Dhruv Mubayi, Douglas B. West. *On the Number of Vertices with Specified Eccentricity* // Graphs Comb. 2000. V.16. P.441-452.

Such a lower bound of the center cardinality is reduced to trivial, and the upper bound, due to the definition, does not take into account possible jumps and gaps of center cardinality values in the interval $\left[\left[1,\ f_{n,k}(\lceil \frac{k}{2} \rceil)\right]\right]$, defined by these estimates, and also turns out to be uninformative for the study of the distribution of the center cardinalities of the almost all graphs, when n tends to infinity.

Center Spectrum of almost all graphs from $\mathcal{J}_{n,\, d=k}$

Further, we asymptotically study the center spectrum of n-vertex graphs of a fixed diameter. It is proved that the center of almost all n-vertex graphs of diameter k

- has cardinality n for $k = 1, 2$, and
- has cardinality $n - 2$ for $k = 3$, while
- for $k \geq 4$ the center spectrum is bounded by an interval of consecutive integers and additionally contains at most one value (two values) outside this interval for even diameter k (for odd diameter k) depending on the value k [25].

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[25] T.I. Fedoryaeva. *Center and its spectrum of almost all n-vertex graphs of given diameter* // Siber. Electr. Math. Reports. 2021. V.18. P.511–529.

Theorem 14 (spectrum of almost all graphs from $\mathcal{J}_{n,\,d=k}$[25]).

Let $k \geq 1$ and $p \geq 1$ be fixed integer constants. Then
(i) $|\mathbb{C}(G)| = n$ *for almost all n-vertex graphs G of diameter* $k = 1, 2$;
(ii) $|\mathbb{C}(G)| = n - 2$ *for almost all n-vertex graphs G of diameter* 3;
(iii) $|\mathbb{C}(G)| \in \big[\big[\,1 + p,\ n - 5 - p\,\big]\big]$ *for almost all n-vertex graphs G of diameter* 4;
(iv) $|\mathbb{C}(G)| \in \big[\big[\,2 + p,\ n - 5 - p\,\big]\big] \cup \{n - 4\}$ *for almost all n-vertex graphs G of diameter* 5; *moreover, the fraction of such graphs with an $(n-4)$-vertex center asymptotically equals* $\frac{1}{3}$;

Theorem 14.

(v) $|\mathbb{C}(G)| \in \{1\} \cup \left[\left[\, 1+p,\ n-k-1-p \,\right]\right]$ *for almost all n-vertex graphs G of even fixed diameter* $k \geq 6$; *moreover, the fraction of such graphs with a trivial center asymptotically equals* $\frac{k-4}{k-2}$;

(vi) $|\mathbb{C}(G)| \in \{2\} \cup \left[\left[\, 2+p,\ n-k-p \,\right]\right] \cup \{n-k+1\}$ *for almost all n-vertex graphs G of odd fixed diameter* $k \geq 7$; *moreover, the fraction of such graphs with a 2-vertex and an* $(n-k+1)$*-vertex center asymptotically equals* $\frac{k-5}{k-2}$ *and* $\frac{1}{k-2}$ *respectively.*

Note that the realizability of the cardinalities spectrum of the center found in Theorem 14 is established in the class of constructed typical graphs $\mathcal{F}_{n,k,p}$; for this, two constructions for designing such graphs are given (see Theorem 5 in [25]).

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[25] T.I. Fedoryaeva. *Center and its spectrum of almost all n-vertex graphs of given diameter* // Siber. Electr. Math. Reports. 2021. V.18. P.511–529.

Realizability of the found spectrum of center cardinalities

Theorem 15 (spectrum $\mathbb{S}p_c(\mathcal{F}_{n,k,p})$) [25].

Let $k \geq 3$ and $p \geq 1$. Then for every $n \geq 2p + k + 4$ the following equalities hold

(i) $\mathbb{S}p_c(\mathcal{F}_{n,k=3,p}) = \{n-2\}$;

(ii) $\mathbb{S}p_c(\mathcal{F}_{n,k=4,p}) = [[1+p,\, n-5-p]]$;

(iii) $\mathbb{S}p_c(\mathcal{F}_{n,k=5,p}) = [[2+p,\, n-5-p]] \cup \{n-4\}$;

(iv) $\mathbb{S}p_c(\mathcal{F}_{n,k,p}) = \{1\} \cup [[1+p,\, n-k-1-p]]$ *for every even* $k \geq 6$;

(v) $\mathbb{S}p_c(\mathcal{F}_{n,k,p}) = \{2\} \cup [[2+p,\, n-k-p]] \cup \{n-k+1\}$ *for every odd* $k \geq 7$.

[25] T.I. Fedoryaeva. *Center and its spectrum of almost all n-vertex graphs of given diameter* // Siber. Electr. Math. Reports. 2021. V.18. P.511–529.

Properties of the centers

Theorem 14 implies a number of properties of the centers of almost all graphs of fixed diameter k.

For example, there are almost no graphs with a trivial center of diameter $k = 2, 4$ and odd diameter k, while for any even $k \geq 6$ this is not true. Similarly, there are almost no graphs with a 2-vertex center of diameter $k = 1, 3, 5$ and even diameter k, however, for every odd $k \geq 7$ this does not hold.

Unexpected is the jump of the center cardinality outside the interval of the consecutive integer values both from above from $n - k - p$ to $n - k + 1$ for odd diameter $k \geq 5$, and from below from $1 + p$ to 1 for even $k \geq 6$ and from $2 + p$ to 2 for odd $k \geq 7$.

Boundaries of the interval

Note that the boundaries of this interval depend on predetermined arbitrary integer p and shrink when choosing a greater value p. Wherein, graphs whose center cardinality belongs to this interval (or, respectively, coincides with the value outside it) have the non-zero asymptotic fraction.

And typical graphs for the classes of graphs corresponding to these cases of cardinality of the center from Theorem 14 were also constructed in [25].

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[25] T.I. Fedoryaeva. *Center and its spectrum of almost all n-vertex graphs of given diameter* // Siber. Electr. Math. Reports. 2021. V.18. P.511–529.

All obtained typical properties of the center and its spectrum for n-vertex graphs of fixed diameter $k \geq 2$ remain typical for connected graphs of diameter at least k, as well as for graphs (not necessarily connected) with a shortest path of length at least k. In particular, the next corollary is valid.

Corollary 8.

For every fixed $k \geq 2$, almost all n-vertex graphs of each of the following classes $\mathcal{J}_{n,\, d\geq k}$, $\mathcal{J}^*_{n,\, d\geq k}$ are connected, have diameter k, and their center satisfies the properties stated in Theorem 14.

Thank you for your attention!