Rigidity of Riemannian embeddings of discrete metric spaces

Matan Eilat

Weizmann Institute of Science

Workshop on the discrete geometry and the geometry of numbers

Moscow State University
December 2025

Isometric embeddings

- $M = M^n$ - complete, connected, n-dimensional Riemannian manifold.
- Metric: $d_M(x,y) = \inf\{\text{length}(\gamma) : \gamma \text{ is a curve connecting } x \text{ and } y\}$.
- Example: $d_{\mathbb{S}^2}(x,-x) = \pi$.
- $\gamma : I \to M$ is a minimizing geodesic if $d(\gamma(s),\gamma(t)) = |s-t|$.
- γ is a geodesic if it is locally distance minimizing.
- Hopf-Rinow: M is complete $\iff$ geodesics extend indefinitely.
- M is complete $\implies$ $p,q \in M$ can be connected by a min. geodesic.
- A map $f : (X,d_X) \to M$ is an isometric embedding if

$$d_M(f(x),f(y)) = d_X(x,y) \qquad \text{for all } x,y \in X.$$

- Write $X \hookrightarrow M$ if such an embedding exists.

Embeddings of finite spaces

- If $|X| = 3$ then $X \hookrightarrow \mathbb{R}^2$.
- $|X| = 4$ w/ all distances 1. Does $X \hookrightarrow \mathbb{R}^2$? No. But $X \hookrightarrow \mathbb{R}^3, r\mathbb{S}^2$.
- Wald('35), Berestovskii ('86): $|X| = 4$ non-branching $\implies X \hookrightarrow S_k^2$.
- Folklore result: $|X| < \infty$ non-branching $\implies X \hookrightarrow M^2$.

What about countable spaces?

✓ $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ (w.r.t. Euclidean distances)

Question

Does $\mathbb{Z}^2 \times \{0\} \cup \{(0,0,1)\} \hookrightarrow M^2$ for some M^2?

Answer

No!

Theorem (E., Klartag)

If $X \hookrightarrow M^2$ for a net $X \subseteq \mathbb{R}^2$, then M^2 is isometric to $\mathbb{R}^2$.

What about dimension $n > 2$?

$X \hookrightarrow M^n$ for a net $X \subseteq \mathbb{R}^n$.

Proposition

All geodesics passing through $p \in X$ are minimizing.

- Connect p to a sequence $X \ni p_m \rightsquigarrow v \in S^{n-1}$ by min. geodesics.
- Obtain complete minimizing geodesic $\gamma_{p,v}$ in a "global direction" v.
- The map $S^{n-1} \ni v \mapsto \dot{\gamma}_{p,v}(0) \in S_pM$ is odd, continuous and onto.
- The exponential map $\exp_p : T_pM \to M$ is a diffeomorphism.

Theorem (E., Klartag)

M^n is diffeomorphic to $\mathbb{R}^n$.

No conjugate points

Theorem 1 (E. '25)

All geodesics in M are minimizing, and there are no conjugate points in M.

- The "ideal boundary": for $v \in S^{n-1}$ define

$$\partial_v M = \{B : M \to \mathbb{R} : B \text{ is } 1-\text{Lipschitz and } B|_X = \langle \cdot , v \rangle\}.$$

- $B_v(x) = \inf\{B(x) : B \in \partial_v M\}$ induces a foliation by transport lines.
- Want: $S^{n-1} \ni v \mapsto \nabla B_v(x) \in S_x M$ is odd, continuous and onto.
- $\partial_v M$ is a singleton.

No conjugate points

Theorem (Bangert, Emmerich 2013)

Suppose M^2 is s.t. all geodesics are minimizing. Then for any $x \in M$

$$\liminf_{r \to \infty} \frac{\text{Area}(D(x,r))}{\pi r^2} \geq 1,$$

with equality if and only if M is flat.

Theorem (Hopf 1948)

A 2-dimensional torus without conjugate points is flat.

Theorem (Burago, Ivanov 1994)

An n-dimensional torus without conjugate points is flat.

Large-scale geometry

Theorem 2 (E. '25)

X is a net with respect to the Riemannian distance in M.

- There are no mesoscopic portions oblivious of the embedding.

Corollary 1

The map $S^{n-1} \ni v \mapsto \nabla B_v(x) \in S_x M$ is a homeomorphism.

Corollary 2

Let $x \in M$. Then for $a, b \in \mathbb{R}^n$, writing $a = tv, b = sw$ with $v, w \in S^{n-1}$ and $t, s \geq 0$, we have

$$\lim_{r\to\infty} \frac{d(\gamma_{x,v}(tr), \gamma_{x,w}(sr))}{r} = |a - b|,$$

and the convergence is locally uniform in $a, b \in \mathbb{R}^n$.

Thank you!

Questions?