Lecture Notes
Advanced Gas Dynamics
by K.V. Krasnobaev, E.I. Mogilevskiy

Contents:

– conservation laws, examples of complete systems of equations;
– steady flow in ducts, examples of applied gas dynamics;
– one-dimensional wave motions, waves in compressible gas;
– two-dimensional steady flows, slender body linear theory;
– supersonic flows around blunt bodies;
– complicated media models.

References

Russian:

1. *Кочин Н.Е., Кибель И. А., Розе Н. В.* Теоретическая гидромеханика: В 2-х ч./Под ред. И. А. Кибеля. - М.: Физматгиз, 1963.

2. *Ландау Л. Д., Лифшиц Е. М.* Теоретическая физика: В 10-ти т. Т. VI. Гидродинамика. - М.: Наука, 1986.

3. *Черный Г. Г.* Газовая динамика. - М.: Наука, 1988.

4. *Краснобаев К.В.* Лекции по основам механики сплошной среды - М.: Физматгиз, 2005.

English:

1. *Lamb H.* Hydrodynamics 6th edition, Cambridge Univ. Press. 1975

2. *Liepmann H.W., Roshko A.* Elements of Gas Dynamics Dover Publications, 1957

3. *Landau L.D., Lifshitz* Theoretical physics: Vol. 6. Hydromechanics. Pergamon Press, 2nd edition, 1987.

Содержание

1 Basis equations

1.1 Mass, momentum, and energy conservation laws

Consider basic equations governing motion of a medium. These equations mathematically formulate laws based on experimental data. They are mass, momentum, and angular momentum conservation laws, first and second laws of thermodynamics. We postulate that these laws are valid for continuous media.

Derivation of the equations involves considering motion of a material element τ bounded by a surface Σ. For any characteristic of the media $f(t, x, y, z)$ in τ, Reynolds transport theorem (the formula for time derivative of an integral over material volume) reads

$$\frac{d}{dt}\int\limits_{\tau} f\,d\tau = \int\limits_{\tau} \frac{\partial f}{\partial t}\,d\tau + \int\limits_{\Sigma} f v_n\,d\Sigma$$

(v_n is the projection of the velocity vector $\mathbf{v}$ to the external normal vector $\mathbf{n}$ of the surface Σ).

If function f and velocity field $\mathbf{v}$ are smooth enough, the divergence theorem gives

$$\frac{d}{dt}\int\limits_{\tau} f\,d\tau = \int\limits_{\tau} \left(\frac{\partial f}{\partial t} + \frac{\partial f v_x}{\partial x} + \frac{\partial f v_y}{\partial y} + \frac{\partial f v_z}{\partial z}\right) d\tau \qquad (1.1.1)$$

Consider a moving volume τ^* bounded by a surface Σ. Denote velocity of the surface points by N. For any function $A(t, x, y, z)$ we rewrite Reynolds transport theorem:

$$\frac{d}{dt}\int\limits_{\tau^*} A\,d\tau = \int\limits_{\tau^*} \frac{\partial A}{\partial t}\,d\tau + \int\limits_{\Sigma} AN\,d\Sigma\,.$$

If the volume τ^* coincides with a material volume τ at $t = 0$, we have

$$\frac{d}{dt}\int\limits_{\tau} A\,d\tau = \frac{d}{dt}\int\limits_{\tau^*} A\,d\tau + \int\limits_{\Sigma} A(v_n - N)\,d\Sigma\,.$$

Continuity equation. Mass of fluid in the element τ is conserved. Since (1.1.1) gives

$$\frac{d}{dt}\int\limits_{\tau} \rho\,d\tau = \int\limits_{\tau} \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right) d\tau = 0 \qquad (1.1.2)$$

(ρ is the gas density).

As considered volume τ is arbitrary

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} = 0 \qquad (1.1.3)$$

The equations (1.1.2) and (1.1.3) show the mass conservation law in integral and differential forms, respectively.

Momentum equation. By definition, momentum $\mathbf{Q}$ of material volume τ is

$$\mathbf{Q} = \int_\tau \rho \mathbf{v}\, d\tau$$

By Newton's law, change of momentum $d\mathbf{Q}$ over time interval dt equals impulse of external forces acting on all particles inside the τ:

$$d\mathbf{Q} = \left(\int_\tau \rho \mathbf{F}\, d\tau + \int_\Sigma \mathbf{p}_n d\Sigma \right) dt$$

where $\mathbf{F}$ is the mass (body) force density, and $\mathbf{p}$ is the stress vector. For the momentum derivative w.r.t time

$$\frac{d\mathbf{Q}}{dt} = \int_\tau \rho \mathbf{F}\, d\tau + \int_\Sigma \mathbf{p}_n d\Sigma$$

Applying (1.1.1) to the left-hand-side of this equation and taking into account continuity equation, we obtain integral equation of momentum conservation.

$$\int_\tau \rho \frac{d\mathbf{v}}{dt}\, d\tau = \int_\tau \rho \mathbf{F}\, d\tau + \int_\Sigma \mathbf{p}_n d\Sigma \qquad (1.1.4)$$

Cauchy formula reads

$$\mathbf{p}_n = \mathbf{p}_x \cos(\mathbf{n}, x) + \mathbf{p}_y \cos(\mathbf{n}, y) + \mathbf{p}_z \cos(\mathbf{n}, z),$$

and allows introducing of the stress tensor

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{F} + \operatorname{div} \Pi \qquad (1.1.5)$$

$$\operatorname{div} \Pi = \frac{\partial \mathbf{p}_x}{\partial x} + \frac{\partial \mathbf{p}_y}{\partial y} + \frac{\partial \mathbf{p}_z}{\partial z}$$

Equations (1.1.4) and (1.1.5) shows the momentum conservation law in integral and differential forms, respectively.

For classical definition of angular momentum **K**

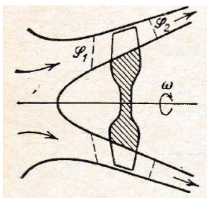
$$\mathbf{K} = \int\limits_{\tau} \mathbf{r} \times \mathbf{v}\rho d\tau$$

the equation (1.1.4) gives angular momentum conservation law:

$$\frac{d\mathbf{K}}{dt} = \int\limits_{\tau} \mathbf{r} \times \mathbf{F}\rho d\tau + \int\limits_{\Sigma} \mathbf{r} \times \mathbf{p}_n d\Sigma \tag{1.1.6}$$

This law is valid for media without angular momentum corresponding to internal degrees of freedom.

Example. Blade machines Blade machines transform mechanical energy of flowing gas to kinetic energy of moving boundary (turbine) or vice versa use rotation of a rigid body for gas acceleration. Gas goes through an annular duct and interacts with blades - some plates placed across the duct. Blades are connected to a massive body in the center and they form a rotor all together. (pic. 1).



Pic. 1: Scheme of a blade machine

Our aim is to find power of a turbine for given flow parameters.

Let ω be the angular velocity of the rotor. Consider a control volume τ^* bounded by the inlet Σ_1, outlet Σ_2 and the walls of the duct (pic. 1). This volume rotates with the rotor. According (1.1.6) the angular momentum conservation law reads

$$\frac{d}{dt}\int_{\tau^*} (\mathbf{r} \times \rho\mathbf{v})d\tau + \int_{\Sigma} (\mathbf{r} \times \rho\mathbf{v})(v_n - D)d\Sigma = \vec{\mathcal{M}}$$

$\vec{\mathcal{M}}$ is the principal torque, $\mathbf{r}$ is the position vector of some point on the duct axis, D is the normal velocity of the surface. The parameters distribution over Σ_1 are axially symmetric and the flow is steady relative to the control volume. So

$$\mathcal{M} = \int_{\Sigma} (\mathbf{r} \times \rho\mathbf{v})(v_n - D)d\Sigma$$

Tangential stresses on the duct walls are negligible and normal stresses give zero torque about the axis of the duct. Project this equation on the axis direction

$$\mathcal{M} = \int_{\Sigma_1+\Sigma_2} r\rho c_u v_n d\Sigma$$

Gas flux though boundaries is proportional to $v_n - D$ and vanishes at walls, so only integral over $(\Sigma_1 + \Sigma_2)$ is non-zero. Scalar r is distance to the axis, c_u is a transversal (circumferential, azimuthal) component of the absolute gas velocity.

Power W of the rotor is

$$W = \omega \mathcal{M} = \omega \int_{\Sigma_1+\Sigma_2} r\rho c_u v_n d\Sigma \tag{1.1.7}$$

or

$$W = \omega \int_0^G [(rc_u)_2 - (rc_u)_1] dm \tag{1.1.8}$$

Here G is the total mass flux, dm is mass of the gas flowing through an element of cross-sections $d\Sigma_1$ and $d\Sigma_2$ of surfaces Σ_1 and Σ_2.

Due to mass conservation

$$dm = \rho_1 v_{n1} d\Sigma_1 = \rho_2 v_{n2} d\Sigma_1$$

and (1.1.8) has a form:

$$W = \omega[(rc_u)_2 - (rc_u)_1]^* G \tag{1.1.9}$$

The $(*)$ means mass average. The formula (1.1.9) was first obtained by Leonard Euler.

The first law of thermodynamics. There exist a variable of state $\mathcal{E}$ such that

$$d\mathcal{E} = dA^{(e)} + dQ^{(e)}$$

where $dA^{(e)}$ is work of external forces over the system and $dQ^{(e)}$ is quantity of heat come from surroundings.

For a material element $d\tau$ with mass of $\rho d\tau$, $\mathcal{E} = (E + U)\,\rho d\tau$, where E is the specific kinetic energy $v^2/2$ and U is a variable of sate, namely the specific internal energy.

The momentum equation gives a value of a kinetic energy change, and the first law of thermodynamics leads to the equation for internal energy only (the energy equation):

$$\frac{dU}{dt} = \frac{1}{\rho}\left(\mathbf{p}_x \frac{\partial \mathbf{v}}{\partial x} + \mathbf{p}_y \frac{\partial \mathbf{v}}{\partial y} + \mathbf{p}_z \frac{\partial \mathbf{v}}{\partial z}\right) + q^{(e)} \tag{1.1.10}$$

where

$$q^{(e)} = \lim_{\substack{d\tau \to 0 \\ dt \to 0}} \frac{dQ^{(e)}}{\rho d\tau dt}$$

is the non-mechanical energy flux from surroundings to a unit mass of medium per unit time.

The second law of thermodynamics. There exist a variable of state S (entropy) such that for reversible processes

$$TdS = dQ^{(e)}.$$

For an irreversible process leading from a state A to a state B, the second law reads

$$S(B) - S(A) \geq \int\limits_{\mathcal{L}} \frac{dQ^{(e)}}{T}.$$

In the right-hand-side, the integral is taken over the curve $\mathcal{L}$ corresponding to the irreversible process in the configuration space. Since entropy is a variable of state $S(A)$ and $S(B)$ are defined independently on the path $\mathcal{L}$.

1.2 Examples of complete systems of equations

One of the main problems of gas dynamics is to derive a complete system of equations. Together with appropriate initial and boundary conditions it allows analysis of particular gas flows.

The most wide-used model is a model of ideal perfect gas. It assumes that all mechanical processes are reversible, the stress tensor is hydrostatic ($\mathbf{p}_n = -p\mathbf{n}, p > 0$), the internal energy U and pressure p depend on two variables of state. The continuity (1.1.3), momentum (1.1.5) and energy (1.1.10) equations reads

$$\begin{aligned} \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0 \\ \rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{F} - \operatorname{grad} p \\ \frac{dU}{dt} + \frac{p}{\rho} \operatorname{div} \mathbf{v} = q^{(e)}. \end{aligned} \tag{1.2.1}$$

Usually $\mathbf{F}$ and $q^{(e)}$ are given functions of variables involved in (1.2.1). As this system of equations is not complete, relations between ρ, p, U (equations of state) must be added.

Thermic $p = p(\rho, T)$ (T being the temperature) and calorific $U = U(\rho, T)$ equations of state are usually given. The functions $p(\rho, T)$ and $U(\rho, T)$ are not independent as they must admit laws of thermodynamics.

Let the density and specific entropy be independent variables, then

$$U = U(\rho, s),\; p = p(\rho, s),\; T = T(\rho, s)$$

are known functions and these relations close (1.2.1). It becomes a complete system of equations for unknowns: $\rho, \mathbf{v}, s$

$$\begin{aligned} \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0 \\ \rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{F} - \operatorname{grad} p \\ T\frac{ds}{dt} = q^{(e)}\,. \end{aligned} \tag{1.2.2}$$

We find p and T by relations

$$p = \rho^2 \frac{\partial U}{\partial \rho}\,, T = \frac{\partial U}{\partial s}$$

The model of perfect gas reads

$$U = c_v T, \quad p = \rho R T, \quad c_v,\, R = \text{const.}$$

Hence specific entropy is

$$s = c_v \ln \frac{p}{\rho^\gamma} + s_0, \quad s_0 = \text{const}, \quad \gamma = \frac{c_v + R}{c_v}$$

Introducing this expression in (1.2.2) simplifies the equations and gives equations for $\rho, \mathbf{v}, T$

$$\begin{aligned} \frac{d\rho}{dt} + \rho\mathbf{v}\, div\, \mathbf{v} = 0 \\ \rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{F} - \operatorname{grad} p \\ c_v T \frac{d}{dt} \ln\left(\frac{p}{\rho^\gamma}\right) = q^{(e)} \\ p = \rho R T. \end{aligned} \tag{1.2.3}$$

For application, it is useful to consider gas as barotropic, i.e. $p = \Phi(\rho)$, the function $\Phi(\rho)$ is the same for any elements in considered domain.

2 Lecture 2. Non-steady one- dimensional flows

2.1 Governing equations

A motion is called one-dimensional if all the parameters of the media depend on one coordinate x and, in general, on time t only. If this coordinate is distance to a certain plane the flow is plane (plane waves), if it is distance to a straight line the flow is axisymmterical (cylindrical waves), and if it is distance to a certain point the flow is spherically symmetric (spherical waves). Assume, that only x component of velocity is non-zero. Body force also has only x non-zero component, if any.

For Eulerian description (1.2.2), unknown function are velocity component u and two thermodynamic variables, e.g. pressure p and density ρ, while x and t are independents variables.

The continuity and motion equations reads

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + (\nu - 1)\frac{\rho u}{x} = 0 \tag{2.1.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = F_x \tag{2.1.2}$$

where $\nu = 1, 2, 3$ used for plane, cylindrical, and spherical waves, respectively.

The entropy change equation is added to the systems of equations (2.1.1), (2.1.2)

$$T\left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x}\right) = q^{(e)} \tag{2.1.3}$$

and a relation for entropy (equation of state) as well

$$p = p(\rho, s), \quad T = T(\rho, s) \tag{2.1.4}$$

Then, the obtained system of equation (2.1.1) — (2.1.3) with additional relation (2.1.4) forms complete system of equations for functions $u(x,t)$, $\rho(x,t)$, $p(x,t)$.

Further, we will use the continuity equation (2.1.1) in the form containing full derivative of p over time. Since $\rho = \rho(p, s)$, introducing speed of sound a as $a^2 = (\partial p/\partial \rho)_s$, from (2.1.1) we have

$$\frac{1}{a^2}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) + \rho\frac{\partial u}{\partial x} + (\nu - 1)\frac{\rho u}{x} = 0 \tag{2.1.5}$$

The system (2.1.1) — (2.1.3) is a PDE system. So, to complete the statement of the problem and find a solution, initial and boundary conditions must be specified.

2.2 Small disturbances

The simplest solution for the gas dynamics equations (2.1.1) — (2.1.3) describes gas in rest:

$$p = p_0, \quad \rho = \rho_0, \quad u = 0, \quad s = s_0, \quad T = T(\rho_0, s_0)$$

p_0, ρ_0, s_0 are constants.

In this lecture we analyse solutions which are close to this one.

Consider 1D non-stationary flows, where pressure p, density ρ slightly differ from constant values p_0, ρ_0, and gas velocity u is small. The scale for velocity comes from the gas state, it is speed of sound $a_0 = \sqrt{(\partial p/\partial \rho)_s}$. This velocity has the same order of magnitude as velocity of chaotic motion of gas molecules.

Disturbances are small so $p \ll p_0$, $\rho \ll \rho_0$, $u \ll a_0$, $s - s_0 \ll s_0$ and smooth $\partial/\partial t \ll a_0 \partial/\partial x$. Nonlinear terms in governing equations: continuity (2.1.1), equation of motion (2.1.2) and entropy (2.1.3) can be omitted and these equations read

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} + (\nu - 1)\frac{\rho_0 u}{x} = 0 \tag{2.2.1}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0 \tag{2.2.2}$$

$$\frac{\partial s}{\partial t} = 0 \tag{2.2.3}$$

As $\rho = \rho(p, s)$, we have

$$\rho - \rho_0 = \left.\frac{\partial \rho}{\partial p}\right|_s (p - p_0) + \left.\frac{\partial \rho}{\partial s}\right|_p s \tag{2.2.4}$$

Potential φ integrates (2.2.2):

$$u = \frac{\partial \varphi}{\partial x}, \quad p - p_0 = -\rho_0 \frac{\partial \varphi}{\partial t} \tag{2.2.5}$$

Differentiation of (2.2.4) w.r.t. time according (2.2.3) and (2.2.5) leads to equation for φ from (2.2.1):

$$\frac{1}{a_0^2}\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\nu - 1}{x}\frac{\partial \varphi}{\partial x} = 0 \tag{2.2.6}$$

Plane waves. For $\nu = 1$, (2.2.6) is a classical wave equation and its general solution is

$$\begin{aligned}
&\varphi = F(x - a_0 t) + G(x + a_0 t) \\
&u = f(x - a_0 t) + g(x + a_0 t) \\
&p - p_0 = \rho_0 a_0 \left[f(x - a_0 t) - g(x + a_0 t) \right] \\
&f = F', \quad g = G'
\end{aligned}$$

Values $r = u + (p - p_0)/(\rho_0 a_0)$ and $l = u - (p - p_0)/(\rho_0 a_0)$ are constant on right-running and left-running characteristics $x = x_0 \pm a_0 t$, respectively. This form of the solution simplifies analysis of reflection.

If a single pulse appears at $t = 0$ at some finite segment, waves go to both direction and their shape does not change during the propagation.

Spherical waves. For $\nu = 3$, (2.2.6) is equivalent to

$$\frac{1}{a_0^2} \frac{\partial^2 x\varphi}{\partial t^2} - \frac{\partial^2 x\varphi}{\partial x^2} = 0$$

It is again a wave equation, so its general solution has a form:

$$\varphi = \frac{1}{x} \left[F(x - a_0 t) + G(x + a_0 t) \right]$$

Velocity and pressure are

$$\begin{aligned}
&u = \frac{f(x - a_0 t)}{x} - \frac{1}{x^2} \int_{\xi_0}^{x - a_0 t} f(\xi) d\xi + \frac{g(x + a_0 t)}{x} - \frac{1}{x^2} \int_{\eta_0}^{x + a_0 t} g(\eta) d\eta \\
&\frac{p - p_0}{\rho_0 a_0} = \frac{f(x - a_0 t)}{x} - \frac{g(x + a_0 t)}{x}
\end{aligned}$$

Consider a particular solution describing pressure wave propagation away from the center with potential φ

$$\varphi = -\frac{1}{4\pi x} Q \left(t - \frac{x}{a_0} \right) \tag{2.2.7}$$

Velocity and pressure are

$$\begin{aligned}
&u = \frac{\partial \varphi}{\partial x} = \frac{1}{4\pi} \left[\frac{Q' \left(t - \frac{x}{a_0} \right)}{a_0 x} + \frac{Q \left(t - \frac{x}{a_0} \right)}{x^2} \right] \\
&p - p_0 = -\rho_0 \frac{\partial \varphi}{\partial t} = \rho_0 a_0 \frac{1}{4\pi x} Q' \left(t - \frac{x}{a_0} \right)
\end{aligned}$$

The mass flux $q(x,t)$ through a surface $x = \text{const}$ is

$$q = 4\pi x^2 u = \frac{x}{a_0} Q'\left(t - \frac{x}{a_0}\right) + Q\left(t - \frac{x}{a_0}\right)$$

If $x \to 0$

$$\lim_{x\to 0} q(x,t) = Q(t)$$

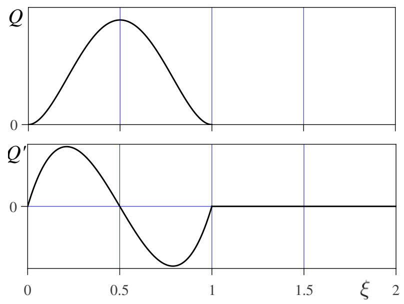
Let the function $Q(t)$ in (2.2.7) is non-zero only if $0 < t < \tau$. Assume, gas comes to rest after the wave is gone. It means

$$Q(0) = Q'(0) = 0,\ Q(\tau) = Q'(\tau) = 0$$

As

$$Q(\tau) = \int_0^\tau Q'(t)dt = 0,$$

the sign of $Q'(t)$ has to change. Consequently, the sign of pressure disturbance changes as well. Note, that there is no such an effect for plane waves.



Pic. 2: Initial disturbance

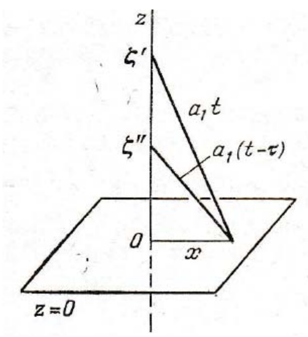
On the other hand, if pressure has a constant sign (or $\int_0^\tau Q'(t)dt = Q(\tau) \neq 0$), gas does not come to rest but moves stationary and has velocity of

$$u = \frac{Q(\tau)}{4\pi x^2}$$

after the wave propagation. Anyway, both pressure and velocity have a factor of $1/x$ and a pulse decreases going from the center. The power of -1 ensures energy conservation.

Cylindrical waves. It is more difficult to derive general solution for cylindrical waves then for plane of spherical ones.
Consider a class of solutions which are represented as a superposition of spherical waves from uniformly distributed over the axis $x = 0$ sources (pic. 3)

$$\varphi = \int \frac{1}{r}\left[F\left(r - a_0 t\right) + G\left(r + a_0 t\right)\right] d\zeta, \qquad (2.2.8)$$
$$r^2 = x^2 + \zeta^2.$$



Pic. 3: Construction of cylindrical wave potential

Consider

$$F = -\frac{1}{4\pi} Q\left(t - \frac{r}{a_0}\right)$$

and find approriate integration limits in (2.2.8)

$$\varphi(t,x) = -\frac{1}{2\pi} \int_0^{\sqrt{a_0^2 t^2 - x^2}} \frac{Q\left(t - \sqrt{x^2 + \zeta^2}/a_0\right)}{\sqrt{x^2 + \zeta^2}} d\zeta \qquad (2.2.9)$$

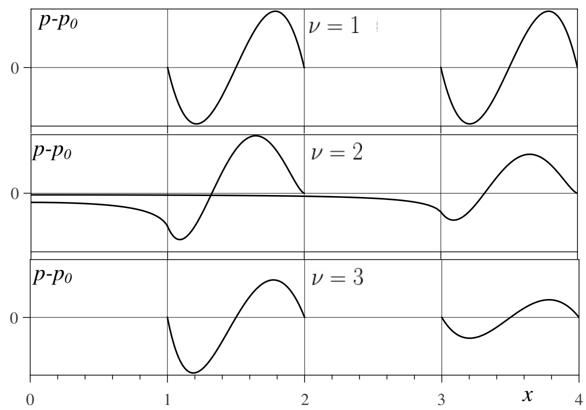
(we took into account that $z = 0$ is a symmetry plane).

If $Q(t)$ is nonzero for a finite time τ, the lower limit in (2.2.9) is actually

$$\zeta_{term} = \sqrt{a_0^2 (t - \tau)^2 - x^2}$$

if this value is greater than zero.

For any fixed $x > 0$, $\varphi \neq 0$ for large enough t: there always is a diapason on the axis which contains sources giving nozero impact at a given point at any instant after first wave come.



Pic. 4: Pressure distribution in impulse for plane, cylindrical, and spherical wave

The pic. 4 shows pressure impulse evolution for plane, cylindrical and spherical cases. Function $F(\xi) = \xi^2(\xi - \tau)^2$ is the same for all geometrical configurations and shown at top left panel and its derivative at bottom left one. This function defines potential. Right panels show pressure distribution in space for two instants: $t = 2\tau$, $t = 4\tau$, the scale for coordinate is $a_0\tau$.

Plane waves propagate without any change. Spherical waves decrease their magnitude while going from the origin. For symmetric in time source, the rear front is steeper than the leading one. Single cylindrical pulse has no rear edge. At any point, pressure goes to zero during infinite time.

3 One-dimensional nonsteady flows

3.1 General properties of characteristics of first-order PDE's with two variables

Consider a system of PDE's with two independent variables. Its general form is

$$(A)f_x + (B)f_t + D = 0 \tag{3.1.1}$$

where (A) and (B) are matrices, D is a vector and their elements depend on x and t, and component of unknown vector-function f. The derivatives f_x, f_t enter the equation (3.1.1) linearly.

The system (2.1.1) — (2.1.3) is an example of such a system: it is linear on derivatives of unknown functions, but its coefficients and free terms depend on unknowns and independent variables arbitrary.

Consider the following problem for the equation (3.1.1). On a certain line $x = x(t)$, values of f are given. It means that the derivative $df/dt = \varphi'(t)$ on this line is also given. Values f_x comes from (3.1.1).

This problem is equivalent to Cauchy problem: find a solution of (3.1.1) such that $f = \varphi(t)$ on $x = x(t)$. If there exists a unique solution, then f_x is determined on $x = x(t)$. In other words, if f_x either doesn't exist or is non-unique, then the Cauchy problem either has no solution or has more than one solution.

To obtain an equation for f_x, we eliminate f_t:

$$\varphi'(t) = \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} \quad f_t = -\frac{dx}{dt}f_x + \varphi'(t) \tag{3.1.2}$$

Substituting f_t from (3.1.2) to (3.1.1) we obtain system of linear algebraic equations for f_x. If the determinant of this system is non-zero, we have a unique solution for f_x on $x = x(t)$. If

$$\Delta = \left|(A) - \frac{dx}{dt}(B)\right| = 0\,, \tag{3.1.3}$$

the system of equations for f_x either has no solution or has infinitely many solutions. In the latter case, the function φ can not be stated arbitrary. Solvability of the system requires certain conditions for φ.

The equation $\Delta = 0$ determines directions dx/dt of characteristics.

Example 1. Find characteristics of (2.1.1) — (2.1.3) with use of (2.1.4). We change (2.1.3) by the equivalent introducing speed of sound $a = (\partial p/\partial \rho)_s$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} - a^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) = 0,$$

Denote

$$f = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}, (A) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ -a^2u & 0 & u \end{pmatrix}, (B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a^2 & 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} (\nu-1)\rho u/x \\ -F_x \\ 0 \end{pmatrix}$$

Denote $dx/dt = \tau$ and obtain equation (3.1.3) for characteristics

$$|(A) - \tau\,(B)| = \begin{vmatrix} u-\tau & \rho & 0 \\ 0 & u-\tau & 1/\rho \\ a^2(-u+\tau) & 0 & u-\tau \end{vmatrix} = 0.$$

or

$$(u-\tau)^3 + a^2(-u+\tau) = 0\,,\ \tau_1 = u\,,\ \tau_{2,3} = u \pm a.$$

As all three value of τ are real and different, the system (2.1.1) —(2.1.3) is hyberbolic by definition. The values of τ correspond to disturbances, which have velocities of u, $u \pm a$.

It is useful to have characteristic view of (2.1.1) —(2.1.3). It contains derivatives of unknown function over characteristic directions τ. To do it, multiply (2.1.1) by a/ρ and add and subtract (2.1.2). For simplicity, we omit the body force F_x and obtain the following

$$\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x} + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] + (\nu-1)\frac{au}{x} = 0 \qquad (3.1.4)$$

$$\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x} - \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right] - (\nu-1)\frac{au}{x} = 0 \qquad (3.1.5)$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial y} = 0 \qquad (3.1.6)$$

Equations (3.1.4) —(3.1.6) give relation between differentials of unknown functions along characteristics:

$$\begin{array}{c} du + \frac{dp}{\rho a} = -(\nu-1)\frac{au}{x}dt,\ dx = (u+a)dt \\ du - \frac{dp}{\rho a} = (\nu-1)\frac{au}{x}dt,\ dx = (u-a)dt \\ ds = 0,\ dx = udt \end{array} \qquad (3.1.7)$$

Introduce new function v and variables r and l by formulae

$$v = \int dp/\rho a, \quad r = u + v, \quad l = u - v. \qquad (3.1.8)$$

Then (3.1.7) gives

$$dr = -(\nu - 1)\frac{au}{x}dt, \; dx = (u + a)dt$$
$$dl = (\nu - 1)\frac{au}{x}dt, \; dx = (u - a)dt$$

Variables r и l are Riemann invariants. For plane flows ($\nu = 1$) they remain constant along right-running and left-running characteristics ($dx = (u + a)dt$ and $dx = (u - a)dt$, respectively) .

Example 2. Find characteristics of (1.2.2) considering steady two dimensional flows: plane and axisymmetric.

The complete system of equations is

$$\begin{aligned}
&\frac{\partial \rho u y^{\nu-1}}{\partial x} + \frac{\partial \rho v y^{\nu-1}}{\partial y} = 0, \\
&\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \rho F_x, \\
&\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \rho F_y, \\
&u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial x} = 0.
\end{aligned} \tag{3.1.9}$$

This system gets the form

$$Af_y + Bf_x + D = 0,$$

after simple transformations and introducing of speed of sound with

$$f = \begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix}, (A) = \begin{pmatrix} vy^{\nu-1} & 0 & \rho y^{\nu-1} & 0 \\ 0 & \rho v & 0 & 0 \\ 0 & 0 & \rho v & 1 \\ -a^2 v & 0 & 0 & v \end{pmatrix}, (B) = \begin{pmatrix} uy^{\nu-1} & \rho y^{\nu-1} & 0 & 0 \\ 0 & \rho u & 0 & 1 \\ 0 & 0 & \rho v & 0 \\ -a^2 u & 0 & 0 & u \end{pmatrix}, ,$$

$$D = \begin{pmatrix} (\nu - 1)\rho v \\ -\rho F_x \\ -\rho F_y \\ 0 \end{pmatrix}$$

Characteristic direction $\tau = dy/dx$ satisfies (3.1.3)

$$
|(A)-\tau\,(B)| = \begin{vmatrix} \xi y^{\nu-1} & -\rho y^{\nu-1}\tau & \rho y^{\nu-1} & 0 \\ 0 & \rho\xi & 0 & -\tau \\ 0 & 0 & \rho\xi & 1 \\ -a^2\xi & 0 & 0 & \xi \end{vmatrix} = 0, \quad \xi = v - u\tau
$$

or

$$
\rho^2 y^{\nu-1}\xi^2\left(\xi^2 - a^2(\tau^2+1)\right) = 0
$$

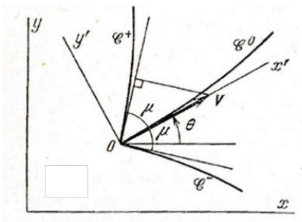
Four roots of this equation are easy to write down:

$$
\tau_{1,2} = \frac{v}{u}, \quad \tau_{3,4} = c_{\pm} = \frac{uv \pm a\sqrt{V^2 - a^2}}{u^2 - a^2}, \quad V^2 = u^2 + v^2.
$$

Two latter ones are real if the flow is supersonic. Characteristic view of the system is more complex and will be obtained by a different way.

The geometry of characteristics is clearly seen in natural coordinate system: x', y', x' being directed along the velocity vector $\mathbf{V}(u', v')$ (or along the entropy characteristic C^0), see fig.5 In this coordinates $v = v' = 0$, $u = u' = V$, so

$$
\left(\frac{dy'}{dx'}\right)_{\pm} = \pm\frac{a}{\sqrt{V^2 - a^2}}
$$



Pic. 5: Geometry of characteristics

It means that characteristics C^+ and C^- makes equal angles with the velocity direction and projection of the velocity vector to normal equals local speed of sound value a. The angle μ between characteristics and velocity vector is called Mach angles, the characteristics are called acoustic or sound. We have

$$
\sin\mu = \frac{a}{V} = \frac{1}{M}, \ \mathrm{tg}\,\mu = \frac{a}{\sqrt{V^2 - a^2}} = \frac{1}{\sqrt{M^2 - 1}}
$$

Or

$$
\left(\frac{dy}{dx}\right)_0 = tg\vartheta, \ \left(\frac{dy}{dx}\right)_{\pm} = tg\,(\vartheta \pm \mu)
$$

4 Two-dimensional steady flows

4.1 Governing equations. First integrals

Consider an ideal perfect gas motion. Assume the flow to be steady, so for unknown functions $\rho, \mathbf{V}, p$ satisfy equations

$$\frac{\partial \mathbf{V}}{\partial t} = 0, \quad \frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial p}{\partial t} = 0$$

Here we consider plane and axisymmetric flows. Gromeka-Lamb equation of motion reads

$$\frac{d\rho}{dt} + \rho \mathrm{div} \mathbf{V} = 0 \tag{4.1.1}$$

$$\nabla \frac{V^2}{2} + 2(\vec{\omega} \times \mathbf{V}) + \frac{1}{\rho} \nabla p = \nabla \mathcal{U} \tag{4.1.2}$$

$$T \frac{ds}{dt} = q. \tag{4.1.3}$$

here ω is vorticity vector, $\mathcal{U}$ is body force potential, $s = s(\rho, p)$ is specific entropy. Specific energy flux q is given.

Projection of (4.1.2) on a streamline $\mathcal{L}$ gives Benoulli integral along a streamline:

$$\frac{V^2}{2} + \int_{p_0}^{p} \frac{dp}{\rho(p, \mathcal{L})} - \mathcal{U} = \mathcal{P}_0(\mathcal{L}) \tag{4.1.4}$$

For barotropic flows, $\rho = \rho(p)$ and the constant $\mathcal{P}_0$ is the same for all streamlines.

Scalar product of (4.1.2) by $\vec{\omega}$ gives the relation along a vortex lines (similar to the streamline case):

$$\frac{V^2}{2} + \int_{p_0}^{p} \frac{dp}{\rho(p, \mathcal{L}^*)} - \mathcal{U} = \mathcal{P}_0(\mathcal{L}^*) \tag{4.1.5}$$

For adiabatic flows, $q = 0$ and (4.1.3) gives one more first integral along streamlines.

$$s(\rho, p) = s(\mathcal{L}) \tag{4.1.6}$$

Equations (4.1.4) — (4.1.6) are first integrals of the system of equations (4.1.1) —(4.1.3). They correspond to two characteristics with $dy/dx = v/u$.

For adiabatic flows, equation (4.1.4) takes form:

$$\frac{V^2}{2} + h - \mathcal{U} = h_0(\mathcal{L}) \tag{4.1.7}$$

Here we use well-known thermodynamic relations $dq = Tds = dU + pd(1/\rho) = dh - dp/\rho$.

So

$$T(\mathbf{V} \cdot \nabla s) = \mathbf{V} \cdot \nabla h - \frac{1}{\rho}(\mathbf{V} \cdot \nabla p)$$

and if $\mathcal{U} = 0$ we have

$$\frac{1}{\rho}\nabla p = \nabla \left(h_0(\mathcal{L}) - \frac{V^2}{2} \right)$$

hence equation of motion (4.1.2) reads:

$$2(\vec{\omega} \times \mathbf{V}) = T\,\nabla s - \nabla h_0 \qquad (4.1.8)$$

This equation is Crocco's vorticity theorem.

Now transform the continuity equation (4.1.1) for adiabatic and barotropic flows in order to eliminate ρ and its derivatives. The barotropy connects density and pressure, and Bernoulli integral (4.1.4) allows calculating substantial derivative of the latter. For simplicity, let $\mathcal{U} = 0$. Hence,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial p}\frac{dp}{dt} = -\frac{\rho}{a^2}\frac{d}{dt}\frac{V^2}{2} = -\frac{\rho}{a^2}\left(\mathbf{V} \cdot \nabla \frac{V^2}{2} \right)$$

The continuity equation has a form:

$$\left(\mathbf{V} \cdot \nabla \frac{V^2}{2} \right) - a^2 \mathrm{div}\mathbf{V} = 0 \qquad (4.1.9)$$

Finally, the system of equations governing steady adiabatic or barotropic motion with no body force is

$$\left(\mathbf{V} \cdot \nabla \frac{V^2}{2} \right) - a^2 \nabla \cdot \mathbf{V} = 0 \qquad (4.1.10)$$

$$\nabla \frac{V^2}{2} + 2(\omega \times \mathbf{V}) + \frac{1}{\rho}\nabla p = 0 \qquad (4.1.11)$$

$$T\frac{ds}{dt} = 0 \qquad (4.1.12)$$

A barotropy equation or a relation between ρ, p, and s closes the system.

4.2 Streamfunction

The continuity equation for two-dimensional flows takes a form:

$$\frac{\partial\, \rho u y^{\nu-1}}{\partial x} + \frac{\partial\, \rho v y^{\nu-1}}{\partial y} = 0$$

with $\nu = 1$ and $\nu = 2$ for plane and axisymmetric cases respectively. Consequently, there exists a scalar function ψ, which differential is $d\psi = \rho u y^{\nu-1} dy - \rho v y^{\nu-1} dx$, and

$$\frac{\partial \psi}{\partial x} = -\rho v y^{\nu-1}, \ \frac{\partial \psi}{\partial y} = \rho u y^{\nu-1}$$

We see that $\psi = \text{const}$ along streamlines $dx/u = dy/v$, and hence (4.1.6), (4.1.7), (4.1.9) take a form ($\mathcal{U} = 0$)

$$\tfrac{u^2+v^2}{2} + h = h_0(\psi) \qquad (4.2.1)$$

$$s = s(\psi) \qquad (4.2.2)$$

$$(a^2 - u^2)\tfrac{\partial u}{\partial x} - uv\left(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\right) + (a^2 - v^2)\tfrac{\partial v}{\partial y} + (\nu - 1)\tfrac{a^2 v}{y} = 0 \qquad (4.2.3)$$

as

$$\left(\mathbf{V} \cdot \nabla \frac{V^2}{2}\right) = u\frac{\partial}{\partial x}\frac{u^2 + v^2}{2} + ... = u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + \dots$$

The equation (4.2.3) involves speed of sound a which depends on thermodynamic parameters h and s. These parameters are functions of ψ and $(u^2 + v^2)$ according to (4.2.1) and (4.2.2).

As h and s actually depend on ψ only, projection of (4.1.8) on y axis gives

$$u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = T\frac{\partial s}{\partial y} - \frac{\partial h_0}{\partial y}$$

and

$$\omega = \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \left(T\frac{ds}{d\psi} - \frac{dh_0}{d\psi}\right)\rho y^{\nu-1} \qquad (4.2.4)$$

Equations (4.1.1) — (4.1.3) give two differential equations along streamlines

$$d\frac{V^2}{2} + \frac{dp}{\rho} = 0, \ ds = 0 \qquad (4.2.5)$$

and two partial differential equations (4.2.3) and (4.2.4). In general, the definition of the streamfunction must be added: $\partial\psi/\partial x = -\rho v y^{\nu-1}$ (or $\partial\psi/\partial y = \rho u y^{\nu-1}$).

Relations (4.2.5) has characteristic form (it contains derivatives along fixed line, namely, the streamline). The Bernoulli integral, entropy, and streamfunction ψ are invariants of these characteristics.

Transform the other equations (4.2.3) and (4.2.4) to characteristic form. Take a sum of (4.2.4) multiplied by a factor λ and (4.2.3):

$$\left(a^2 - u^2\right)\frac{\partial u}{\partial x} - (uv + \lambda)\frac{\partial u}{\partial y} - (uv - \lambda)\frac{\partial v}{\partial x} + \left(a^2 - v^2\right)\frac{\partial v}{\partial y} = \lambda\Omega^* - (\nu - 1)\frac{a^2 v}{y} \qquad (4.2.6)$$

(we denote the rhs of (4.2.4) by Ω^*). This equation gives criteria for combinations of derivatives wrt x and y to be derivatives along a certain direction with slope c. Roots of this equation were already found by general approach for characteristics:

$$\left(\frac{dy}{dx}\right)_{\pm} = c_{\pm} = \frac{uv \pm a\sqrt{V^2 - a^2}}{u^2 - a^2}$$

They are real if $V \geq a$. From (4.2.6), $\lambda_{\pm} = \pm a\sqrt{V^2 - a^2}$.

Equations (4.2.3), (4.2.4) have the following characteristic form:

$$\frac{\partial u}{\partial x} + c_{\pm}\frac{\partial u}{\partial y} + c_{\mp}\left(\frac{\partial v}{\partial x} + c_{\pm}\frac{\partial v}{\partial y}\right) = \frac{1}{u^2 - a^2}\left[(\nu - 1)\frac{a^2 v}{y} - \lambda_{\pm}\Omega^*\right]$$

And characteristic relations are

$$du + c_{-}dv = K_{+}dx \text{ if } dy = c_{+}dx \qquad (4.2.7)$$

$$du + c_{+}dv = K_{-}dx \text{ if } dy = c_{-}dx \qquad (4.2.8)$$

$$\left.\begin{aligned} d\left(\frac{u^2 + v^2}{2}\right) + dh = 0 \\ ds = 0 \\ d\psi = 0 \end{aligned}\right\} \text{if } dy = \frac{v}{u}dx$$

where

$$K_{\pm} = \frac{1}{u^2 - a^2}\left[(\nu - 1)\frac{a^2 v}{y} - \lambda_{\pm}\Omega^*\right]$$

Coming back to initial equations (4.1.1) — (4.1.3), we see that plane and axysimmetrical flows differ by form of the continuity equation. Hence, all results derived for plane flows are valid for axisymmetric ones, if the continuity equation has not been taken into account. In particular, Rankine-Hugoniot relations are valid

$$\rho_1 v_{n1} - \rho_2 v_{n2} = 0$$
$$\rho_1 v_{n1}\mathbf{V}_1 - \rho_2 v_{n2}\mathbf{V}_2 = \mathbf{p}_{n1} - \mathbf{p}_{n2}$$

and shock polar (Busemann curve) has the same form as for $\nu = 1$ (V_c is critical speed)

$$v^2 = (V_1 - u)^2 \frac{u - V_c^2/V_1}{2V_1/(\gamma + 1) + V_c^2/V_1 - u} \qquad (4.2.9)$$

On the other hand, relations on characteristics (4.2.7), (4.2.8) for $\nu = 1$ and $\nu = 2$ are significantly different. Zero right hand side part for plane irrotational flows allows finding the characteristics in godograph plane independently on the solution.

For $\nu = 2$ Bernoulli integral reads

$$\frac{u^2+v^2}{2} + \frac{a^2}{\gamma-1} = \frac{\gamma+1}{\gamma-1}\frac{a_c^2}{2}$$

so solutions for supersonic flow lies between circles $u^2 + v^2 = a_c^2$ and $(\gamma + 1)a_c^2/(\gamma - 1)$ on $u,\ v$ plane. Besides, characteristics could not be found unless the solution is known. This shows formal analogy between plane vortical and axisymmetrical flows.

Equation (4.2.4) indicates the case of transition between potential and rotational plane flows. If a flow is continuous, Kelvin's circulation theorem ensures that vorticity is frozen, and if $\omega = 0$ in some domain, this values will be kept on all streamlines crossing it. Shock waves do not change full enthalpy H_0 so the only possible source of vorticity is non-uniform entropy change at a shock wave. It takes place if the shock is curvilinear. Weak shock waves produce small entropy change, proportional to the third power of the wave intensity, and keep the flow irrotational.

5 Axisymmetric simple waves

5.1 General theory

Consider axisymmetric steady potential flow of a perfect gas with constant adiabatic exponent γ. Let $x,\ y$ be cylindrical coordinates, x goes along the axis of symmetry, y is distance to the axis. We consider self-similar solutions only: they depend on variable $\xi = y/x$. These are simple waves or Busemann flows. In the hodograph plane $u,\ v$, they correspond to curves $v = v(u)$.

Let $r,\ \varphi$ be polar coordinates in a half-plane $y > 0$ $(0 \le \varphi \le \pi)$. The angle φ increases from the positive direction of the x axis. Absence of vorticity condition reads

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$
$$\xi \equiv \frac{y}{x} = tg\,\varphi,\ u = u(\xi),\ v = v(\xi)$$
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x},\ \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y},\ d\xi = \frac{d\xi}{d\varphi}d\varphi$$

Hence

$$\tfrac{dv}{d\varphi} tg\,\varphi + \tfrac{du}{d\varphi} = 0 \tag{5.1.1}$$
$$\tfrac{dv}{du} tg\,\varphi + 1 = 0 \tag{5.1.2}$$

The direction of a ray $\varphi = const$ where velocity components equals (u, v) is normal to curve $v = v(u)$ on the godograph plane.

The continuity equation (4.2.3) and (5.1.1) give

$$\begin{aligned} &N\frac{du}{d\varphi} = a^2 v \\ &N = a^2 - (v\ \cos\ \varphi - u\ \sin\ \varphi)^2,\ \ N = a^2 - v_n^2 \end{aligned} \tag{5.1.3}$$

where $v_n = v\ \cos\ \varphi - u\ \sin\ \varphi$ is a normal to the $\varphi = const$ ray component of velocity.

The derivative of (5.1.2) with expression for $du/d\varphi$ from (5.1.3) gives

$$vv'' = 1 + v'^2 - \frac{(u + vv')^2}{a^2} = \frac{a^2 - v_n^2}{a^2}\left(1 + v'^2\right) \tag{5.1.4}$$

(primes stand for differentiation w.r.t. u).

Each solution of (5.1.4) corresponds to a simple wave. The function $v = v(u)$ gives dependence of u and v on φ by use of (5.1.2). Uniqueness of the solution requires that the integral curve $v(u)$ has no inflection points.

5.2 Supersonic flow past a cone

Consider a supersonic flow past an infinite circular cone with zero angle of attack. The problem has no length scale and is self-similar. The bow shock wave is conical and has equation $\varphi = \varphi_S$. The incoming flow is uniform, the angle between velocity and the shock wave is the same at all points, hence the shock intensity and the entropy change is the constant. After the shock the flow is again isentropic. Full enthalpy does not change on the shock wave. According to Crocco's theorem (4.2.4), vorticiy is zero after the shock wave and the flow obeys (5.1.4) in hodograph plane and (5.1.1) in physical plane.

The boundary conditions for the problem come from Rankine-Hugoniot conditions at the shock wave and non penetration and the cone surface.

Let the cone have semiangle of φ_0. Non-penetration condition reads

$$\frac{v}{u} = tg\,\varphi_0$$

on hodograph plane

$$\frac{v}{u} v' + 1 = 0 \tag{5.2.1}$$

It means that a normal to the integral curve $v = v(u)$ at the corresponding to the cone surface points goes through the origin.

At the shock wave $\varphi = \varphi_S$ velocity components u and v are connected with incoming velocity V_0 via Busemann equation (4.2.9). Finally,

$$\begin{aligned} &\text{at}\,\varphi = \varphi_0: \quad && \frac{v}{u} = tg\,\varphi_0 \\ &\text{at}\,\varphi = \varphi_S: \quad && u + v tg\,\varphi_S = V_0, \quad v = V(u) \end{aligned} \tag{5.2.2}$$

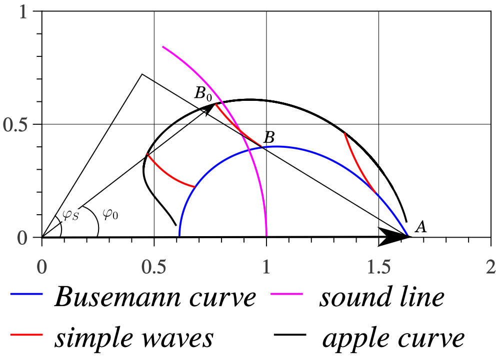
where $V^2(u)$ is right hand side of (4.2.9).

Three boundary conditions (5.2.2) complete the boundary-value problem for the second-order ODE (5.1.4) as the bow shock wave position φ_S is unknown a priopi.

It is more convenient to fix φ_S instead of φ_0 and find the latter. In this manner correspondence between these two angles is stated as well.

We solve this problem using some graphics. First we use Busemann curve (pic.6), draw it for the incoming velocity V_0 (point A). Fix an angle $\varphi = \varphi_S$ and draw an perpendicular from A. This perpendicular crosses the Busemann curve at the point B. It states boundary conditions u and v at $\varphi = \varphi_S$ after the shock wave. The equation (5.1.2) gives direction of the integral curve at the point B:

$$v' tg\, \varphi_S + 1 = 0$$



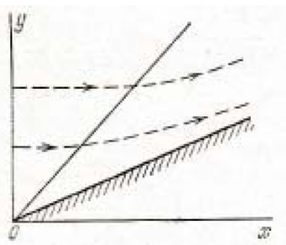
Pic. 6: Towards graphical solution of the flow past a cone problem

The curve is normal to the shock, so it goes along AB.

After the shock wave the normal velocity is subsonic, hence the curve is convex towards the origin. The sign or the curvature corresponds to the sign of v'' (5.1.4). While φ decreases the ray with this direction goes clockwise, so the normal to the integral curve at hodograph plane does. Hence, the integral curve goes to the left from he point B. The curve ends at a point B_0 where normal passes through the origin. The direction and length of OB_0 show the velocity direction and magnitude at the cone surface.

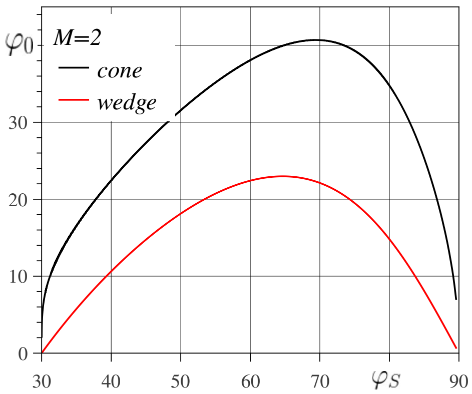
This algorithm can be applied to all possible angles of the shock wave φ_S, ($\mu_1 < \varphi_S < \pi/2$) ($\mu_1$ is a limit angle from Busemann curve). All points B_0 form "apple"curve at the hodograph plane (pic.6).

Qualitative description. After the shock wave the flow is either subsonic or supersonic. Between the shock wave and the cone surface, gas turns further towards the shock wave (pic.7) and Mach number decreases. A transition to subsonic flow is possible, in this case sonic surface is also conical. Opposite to plane case (flow past a wedge), gas has some space after the shock wave to align to the surface, so the maximal angle of an object with attached bow shock is large for cone.

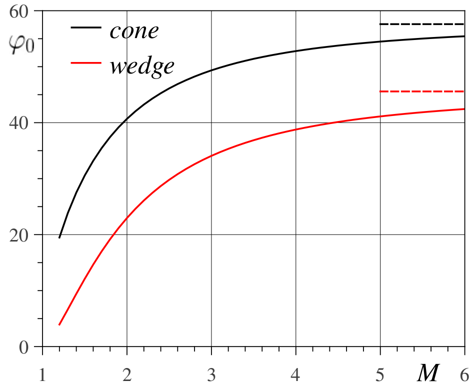


Pic. 7: Streamlines of the flow past a cone

The pic.8 shows dependence of object angle on the shock wave angle for $M = 2$. The difference between lines is the angle of the flow turning after the shock wave. The maximal angles of objects with attached shock wave for cone and wedge are displayed in fig. 9.



Pic. 8: Object angle dependence on shock wave angle



Pic. 9: Maximal object angles dependence on Mach number. Dashed lines correspond to $M \to \infty$

6 Flow past a slender body

6.1 Small perturbation theory

General governing equations for inviscid compressible gas flow are

$$\frac{d\rho}{dt} + \rho \mathrm{div}\mathbf{V} = 0 \tag{6.1.1}$$

$$\mathrm{grad}\frac{V^2}{2} + 2(\omega \times \mathbf{V}) + \frac{1}{\rho}\, grad\, p = \mathrm{grad}\,\mathcal{U} \tag{6.1.2}$$

$$T\frac{ds}{dt} = q \tag{6.1.3}$$

Here ω is vorticity, $\mathcal{U}$ is body force potential, and $s = s(\rho, p)$ is specific entropy. Heat source distribution q is a given function.

These equations have first integrals, namely Bernoulli integral (along a streamline)

$$\frac{V^2}{2} + \int_{p_0}^{p} \frac{dp}{\rho(p, \mathcal{L})} - \mathcal{U} = \mathcal{P}_0(\mathcal{L}) \tag{6.1.4}$$

For baroptopic flows $\rho = \rho(p)$, the constant $\mathcal{P}_0$ is the same for all streamlines. For adiabatic flows $ds = 0$, entropy is also constant along a streamline (6.1.3)

$$s(\rho,\, p) = s(\mathcal{L}) \tag{6.1.5}$$

In this case, the integral in (6.1.4) can be expressed explicitly.

An evident solution of (6.1.1) — (6.1.3) is a uniform flow, which is a flow past a plane with zero angle of attack

$$\mathbf{v} = \mathbf{V}_1,\ p = p_1,\ \rho = \rho_1$$

Consider flows which are close to uniform. For example, these could be flows past a plane with a small topography or a body with surface close to a plane, flows slightly different from the uniform $\mathbf{v} = \mathbf{V}_1$ и $\rho = \rho_1$ at infinity, flows past a weakly oscillating bodies.

We consider flows past a non-moving bodies only. Assume that flow is adiabatic and there is no body forces. Let the gas be a perfect gas with constant heat capacities and the heat capacities ratio is γ.

The incoming flows is uniform. If the flow is subsonic, total enthalpy h_0 and entropy s are constant. Shock waves in supersonic flows cause change of entropy keeping h_0 constant, before the waves both values are constant.

Simplify governing equations due to small magnitude of disturbances.

Bernoulli integral (6.1.4) reads

$$\frac{d}{dt}\frac{V^2}{2}+\frac{1}{\rho}\frac{dp}{dt}=0$$

From continuity equation (6.1.1)

$$\frac{d\rho}{dp}\frac{dp}{dt}+\rho\mathrm{div}\mathbf{v}=0,$$

we have

$$\left(\mathbf{v}\cdot\mathrm{grad}\frac{V^2}{2}\right)-a^2\mathrm{div}\mathbf{v}=0. \tag{6.1.6}$$

Introducing disturbances velocity field

$$\mathbf{v}=(V+u,v,w)$$

make transformations of (6.1.6)

$$\begin{aligned}
&a^2\mathrm{div}\mathbf{v}=a^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=\\
&=(V+u)\frac{\partial}{\partial x}\frac{|\mathbf{v}|^2}{2}+v\frac{\partial}{\partial y}\frac{|\mathbf{v}|^2}{2}+w\frac{\partial}{\partial z}\frac{|\mathbf{v}|^2}{2}=(V+u)^2\frac{\partial u}{\partial x}+v^2\frac{\partial v}{\partial y}+w^2\frac{\partial w}{\partial z}+\\
&+(V+u)v\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+vw\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+(V+u)w\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)
\end{aligned} \tag{6.1.7}$$

Value of local speed of sound a^2 comes from (6.1.4)

$$a^2=a_1^2-(\gamma-1)\left(Vu+\frac{u^2+v^2+w^2}{2}\right) \tag{6.1.8}$$

Let $\varepsilon=u/V$ be a small parameter. It indicates declination of the velocity from mean flow direction $\mathbf{V}_1$. The ratio $v/(V+u)$ is of order of ε as well. Assume, u is of order of ε and velocity disturbances are small compared to a. Then

$$\left(\frac{V}{a}\frac{u}{V}\right)^2\sim\left(M^2\varepsilon^2\right)\ll 1$$

It means, hypersonic flows $M^2\varepsilon^2\geq 1$ are not considered.

Linearization of (6.1.7) gives

$$\left[a^2-(V+u)^2\right]\frac{\partial u}{\partial x}+a^2\frac{\partial v}{\partial y}+a^2\frac{\partial w}{\partial z}=0 \tag{6.1.9}$$

we still keep a term of order of ε^2, this will be explained further.

As possible shock waves are weak, change of entropy is a small values of the third order and can be neglected. It means the flow is isentropic and irrotational due to boundary conditions at infinity.

Combining (6.1.9) and (6.1.8) gives the main equation

$$\left[1 - M^2 - (\gamma + 1)\, M^2 \frac{u}{V}\right] \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \text{ , где } M^2 = V^2/a_1^2 \quad (6.1.10)$$

The only nonlinear term is the last one in square brackets. For transonic flows ($M \approx 1$)it can be the leading one and the equation is sufficiently nonlinear. This term can be omitted,

$$\frac{(\gamma+1)M^2}{|1-M^2|} \frac{|u|_{max}}{V} \ll 1 \quad (6.1.11)$$

In this case, governing equation (6.1.10) reads

$$\left(1 - M^2\right) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (6.1.12)$$

As the flow is irrotational, there exists velocity potential $\varphi(x, y, z)$: $\mathbf{v} = \mathbf{V}_1 + \operatorname{grad}\varphi$. Equations (6.1.10) and (6.1.12) gives for potential

$$\left[1 - M^2 - (\gamma + 1)\, M^2 \frac{1}{V}\frac{\partial \varphi}{\partial x}\right] \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (6.1.13)$$

$$\left(1 - M^2\right) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (6.1.14)$$

Pressure distribution can be found afterwards for given φ. Bernoulli integral reads

$$\frac{(V+u)^2 + v^2 + w^2}{2} + \frac{\gamma}{\gamma - 1}\left(\frac{p}{\rho} - \frac{p_1}{\rho_1}\right) = \frac{V^2}{2} \quad (6.1.15)$$

as entropy is constant and $p = const\, \rho^\gamma$, we have

$$\frac{p - p_1}{\rho_1} = -\left(Vu + \frac{1}{2}(1 - M^2)u^2 + \frac{v^2 + w^2}{2}\right) \quad (6.1.16)$$

6.2 Boundary conditions

Gas does not penetrate into a rigid body. Let the body surface have equation

$$F(x, y, z) = 0 \quad (6.2.1)$$

or

$$y = Y(x, z) \tag{6.2.2}$$

For linear approximation, we have

$$v(x, \pm 0, z) = V \frac{\partial Y}{\partial x} \tag{6.2.3}$$

For body of revolution, we transform boundary condition (6.2.2)

$$F = y^2 + z^2 - R^2(x) = r^2 - \frac{S(x)}{\pi} = 0$$

where r is distance to the axis of symmetry, R and S are radius and normal cross-section area of the body.

Full (nonlinear) boundary condition is

$$-(V + u)\frac{dR}{dx} + v_r = 0$$

Omitting small value of u gives

$$\text{at } \; r = R(x): \;\; r v_r = V R \frac{dR}{dx} \tag{6.2.4}$$

This condition can be simplified by transferring to the axis of symmetry. Taylor expansion gives

$$r v_r = (r v_r)_{r=0} + \left.\frac{\partial r v_r}{\partial r}\right|_{r=0} R(x) + ...$$

The continuity equation gives

$$r \frac{\partial u}{\partial x} = -\frac{\partial r v_r}{\partial r}$$

so the second and further term are small and for the first term, we have

$$(r v_r)_{r=0} = V R \frac{dR}{dx}.$$

This equation can be interpreted as volume source distribution along the axis of symmetry. Their interaction with incoming flow forms a separation surface which is equivalent to a rigid body. The singularity $v_r \to \infty$ as $r \to 0$ is actually inside the rigid body but not in the physical flow domain. At infinity disturbances must vanish, where this condition is applicable. At least, the solution is finite everywhere.

We see one more difference between plane and axisymmetric flows. For the former, longitudinal velocity disturbance u has the same order of magnitude as v. Indeed, from (6.2.3) and irrotationality condition,

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = V\frac{d^2Y}{dx^2}$$

and u has no singularity. Hence it has the same order of magnitude as v throughout the domain.

For axisymmetric flows $v \sim r^{-1}$ as $r \to 0$, and $v_r = a_0 r^{-1} + a_1 + a_2 r + \dots$. Irrotationality condition gives

$$u = a_0' \ln r + a_1' r + a_2' r^2$$

Boundary condition (6.2.4) gives $a_0 = VRR'$ and at the boundary we have

$$u = V\left(R\frac{dR}{dx}\right)' \ln R$$

When $\ln R$ is not large by absolute value $u \sim r/Lv_r$ (L is a lengthscale along x).

For pressure distribution, Bernoulli integral gives (6.1.16)

$$\frac{p - p_1}{\rho_1} = -\left(Vu + \frac{v_r^2}{2}\right) \tag{6.2.5}$$

For plane flows, the first term is leading and pressure coefficient is

$$C_p = \frac{p - p_1}{\rho_1 V^2/2} = -2\frac{u}{V}.$$

Both term in (6.2.5) are sufficient for bodies of revolution and

$$C_p = -2\frac{u}{V} - \left(\frac{v_r}{V}\right)^2.$$

7 Solition for potential

7.1 General solution

General equation for potential of small distubances is (6.1.14). Considering axisymmetric problems in cylindric coordinates x, r, it gives

$$\left(1 - M^2\right) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} = 0 \tag{7.1.1}$$

with boundary condition

$$\text{at } r = R(x): \quad r \frac{\partial \varphi}{\partial r} = rv = VR \frac{dR}{dx} \tag{7.1.2}$$

or

$$(rv)_{r=0} = VR \frac{dR}{dx} \tag{7.1.3}$$

Absence of disturbances at $x \to -\infty$ gives

$$\text{at}\, x \;\to\; -\infty \;\; \varphi \to 0$$

Pressure coefficient is

$$C_p = -\left(\frac{2u}{V} + \frac{v^2}{V}\right) \tag{7.1.4}$$

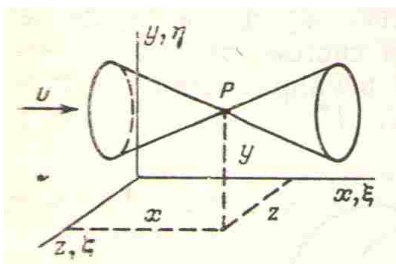
The equation (7.1.1) has different type for subsonuc ($M < 1$) and supersonic ($M > 1$) flows.

If $M < 1$ any disturbance spreads infinitely far upflow and downflow. For $M > 1$ a disturbance lies inside Mach cone with semianle μ of

$$\sin \mu = \frac{a}{V}$$

(a is speed of sound).

For a point P cosider two Mach cones directed upflow and downflow (pic.10). Parameters of the flow at P do not depend on sources of disturbances located outside first cone. On the other hand, a source planced at P does not affect the flow outside the second one.



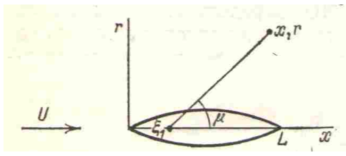
Pic. 10: Mach cones

Boundary condition (7.1.2) sets a distribution of sources along the axis of symmetry. This gives a way to the form of solution of (7.1.1):

$$\varphi = -\int_0^{\xi_1} \frac{q(\xi)\, d\xi}{\sqrt{(x-\xi)^2 + (1-M^2)r^2}} \tag{7.1.5}$$

here ξ is a coordinate on the segment $[0, L]$ of the symmetry axis inside the body, and the density of source intensity is $q(\xi)$ (pic.11).

Consider a point x, r in flow. If $M < 1$, the expression under the root sign is positive for any ξ, and all sources affect the flow in the point. If $M > 1$, the point "feels" the source which have it inside their Mach cone, i.e. $x - \xi \geq \sqrt{M^2-1}\, r$.



Pic. 11

The upper limit in the integral ξ_1 in (7.1.5) is L for $M < 1$ and $\xi_1 = x - \sqrt{M^2-1}\, r$ $(0 < \xi_1 \leq)$ for $M > 1$.

It is convenient to introduce a new variable η as

$$\begin{aligned} x - \xi &= mr \sinh\eta, \quad m = \sqrt{1-M^2} \text{ for } M < 1 \\ x - \xi &= \lambda r \cosh\eta, \quad \lambda = \sqrt{M^2-1} \text{ for } M > 1 \end{aligned} \tag{7.1.6}$$

This rewrites potential as

$$\varphi = \int_{\sinh\eta = x/mr}^{\sinh\eta=(x-L)/mr} q(x - mr\sinh\eta) d\eta \quad \text{for } M < 1 \tag{7.1.7}$$

$$\varphi = \int_{\cosh\eta = x/\lambda r}^{\sinh\eta=0} q(x - \lambda r\cosh\eta) d\eta \quad \text{for } M > 1 \tag{7.1.8}$$

Differentiation gives expressions for velocity components for $M < 1$

$$u = \frac{\partial\varphi}{\partial x} = -\int_0^L \frac{q'(\xi)\, d\xi}{\sqrt{(x-\xi)^2 + m^2r^2}} + \frac{q(L)}{\sqrt{(x-L)^2 + m^2r^2}} - \frac{q(0)}{\sqrt{x^2 + m^2r^2}}$$

$$rv = r\frac{\partial\varphi}{\partial r} = \int_0^L \frac{q'(\xi)(x-\xi)\, d\xi}{\sqrt{(x-\xi)^2 + m^2r^2}} - \frac{q(L)(x-L)}{\sqrt{(x-L)^2 + m^2r^2}} + \frac{q(0)x}{\sqrt{x^2 + m^2r^2}} \tag{7.1.9}$$

and for $M > 1$

$$u = \frac{\partial\varphi}{\partial x} = -\int_0^{\xi=x-\lambda r} \frac{q'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\lambda^2 r^2}} - \frac{q(0)}{\sqrt{x^2-\lambda^2 r^2}}$$
$$rv = r\frac{\partial\varphi}{\partial r} = \int_0^{\xi=x-\lambda r} \frac{q'(\xi)(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2-\lambda^2 r^2}} + \frac{q(0)x}{\sqrt{x^2-\lambda^2 r^2}} \qquad (7.1.10)$$

Point bodies (with $dS/dx = 0$, S being cross-section area) require $q(0) = q(L) = 0$ (subsonc flow) or just $q(0) = 0$ (supersonic flow).

Boundary condition (7.1.2) gives an integral equations for $q(\xi)$:

$$VR\frac{dR}{dx} = \left[\int_0^L \frac{q'(\xi)(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2+m^2r^2}}\right]_{r=R(x)} \quad \text{for } M<1, \quad m=\sqrt{1-M^2}$$

$$VR\frac{dR}{dx} = \left[\int_0^{x-\lambda r} \frac{q'(\xi)(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2-\lambda^2r^2}}\right]_{r=R(x)} \quad \text{for } M>1, \quad \lambda=\sqrt{M^2-1}$$

These equations, again, have different type. The former is Fredholm first kind equation and the latter is Volterra first kind equation. One usually have to solve them numerically.

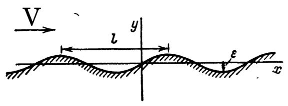
After velocity is known, pressure distribution comes from (7.1.4). The second (quadratic) term is sufficinet. Equations (7.1.9) and (7.1.10) show that $rv \sim uL$, i.e $u \sim rv/L \ll 1$.

Often, the inverse problem is used. One finds potential $\varphi(x, r)$ for given source distribution. Afterwards a suitable rigid surface can be found.

7.2 Examples

Plane flow past a wavy wall. Subsonic flow
Cosider a flow past a plane with sinusoidal topography shown on pic.12. The surface has equation

$$y = Y(x) = \varepsilon \sin \alpha x \qquad (7.2.1)$$



Pic. 12

The value $\varepsilon = 0$ correponds to basic flowwhich is a uniform flow with velocity of V.

Potentail of disturbances obeys the equation (6.1.14)

$$\left(1-M^2\right)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \qquad (7.2.2)$$

The boundary condition (6.2.3) for φ is

$$v_{y=0} = \frac{\partial\varphi}{\partial y} = V\frac{dY}{dx} = V\varepsilon\alpha\,\cos\alpha x \qquad (7.2.3)$$

As the wall is infinite, the velocity is bounded at infinity $u = \partial\varphi/\partial x$ $v = \partial\varphi/\partial y$ at $y \to \infty$.

Let $M^2 < 1$. We use Fourier method solving (7.2.2)

$$\varphi = F(x)G(y)$$

For F and G we have

$$\frac{F''}{F} = \frac{-G''}{(1-M^2)G} = -\lambda^2$$

Value of λ is real and positive as the solution is periodic on x.

Hence

$$F = A\sin\lambda x + B\cos\lambda x$$
$$G = A_1\exp(-\sqrt{1-M^2}\lambda y) + B_1\exp(\sqrt{1-M^2}\lambda y)$$

Boundness at $y \to \infty$ requires $B_1 = 0$, and (7.2.3) gives

$$A = 0, \;\; \lambda = \alpha, \;\; -A_1 B\sqrt{1-M^2} = V\varepsilon$$

The solution is

$$\varphi = -\frac{V\varepsilon}{\sqrt{1-M^2}}\exp(-y\alpha\sqrt{1-M^2})\cos\alpha x$$

Velocity componets and pressure are

$$u = \frac{V\varepsilon\alpha}{\sqrt{1-M^2}}\exp(-y\alpha\sqrt{1-M^2})\sin\alpha x$$
$$v = V\varepsilon\alpha\exp(-y\alpha\sqrt{1-M^2})\cos\alpha x$$
$$p - p_1 = \triangle p = -\frac{\rho_1 V^2\varepsilon\alpha}{\sqrt{1-M^2}}\exp(-y\alpha\sqrt{1-M^2})\sin\alpha x$$

Disturbances have maximal magnitude at the wall and graually decay going from it. It is eqsy to show that drag force is zero.

Linear theory is valid if

$$\frac{u}{V} \ll 1, \; \frac{v}{V} \ll 1, \;\; \frac{(\gamma+1)M^2}{|1-M^2|}\frac{|u|_{max}}{V} \ll 1.$$

For this particular problem it means

$$\frac{\varepsilon\alpha}{\sqrt{1-M^2}} \ll 1, \;\; \frac{(\gamma+1)M^2\varepsilon\alpha}{(1-M^2)^{3/2}} \ll 1$$

The second condition is stronger and more restricitve for the wall steepness $\varepsilon\alpha$.

Plane flow past a wavy wall. Supersonic flow Let $M > 1$. General solution of (7.2.2) is

$$\varphi = F(x - \sqrt{M^2 - 1}y) + G(x + \sqrt{M^2 - 1}y)$$

Characteristics of (7.2.2 are straight lines

$$x - \sqrt{M^2 - 1}y = const, \quad x + \sqrt{M^2 - 1}y = const.$$

Functions F and G are constant at this lines, respectively. As no disturbances come from infinity, $G = 0$.

Boundary condition at the wall gives

$$v = \frac{\partial \varphi}{\partial y} = -\sqrt{M^2 - 1}F'(x) = V\frac{dY}{dx} = V\varepsilon\alpha \cos \alpha x$$

and F is

$$F(x) = -\frac{V\varepsilon}{\sqrt{M^2 - 1}} \sin \alpha x$$

Hence,

$$\varphi = -\frac{V\varepsilon}{\sqrt{M^2 - 1}} \sin \left[\alpha \left(x - \sqrt{M^2 - 1}y\right)\right]$$
$$u = -\frac{V\varepsilon\alpha}{\sqrt{M^2 - 1}} \cos \left[\alpha \left(x - \sqrt{M^2 - 1}y\right)\right]$$
$$v = V\varepsilon\alpha \cos \left[\alpha \left(x - \sqrt{M^2 - 1}y\right)\right]$$
$$\triangle p = \frac{\rho_1 V^2 \varepsilon\alpha}{\sqrt{M^2 - 1}} \cos \left[\alpha \left(x - \sqrt{M^2 - 1}y\right)\right]$$

In supersonic flow, disturbances do not decay but keep constant value along characteristics $x - \sqrt{M^2 - 1}y = const$. Wavy drag force apprears. The force per period is

$$X = \int_0^l \triangle p \frac{dY}{dx} dx = \frac{\rho_1 V^2}{\sqrt{M^2 - 1}} \int_0^l (\varepsilon\alpha \cos \alpha x)^2 dx$$

Physically, it means energy tranfer by acoustic waves.

Supersonic flow past a slender cone Consider an axisymmetric supersonic flow with small disturbances. Set a linear source distribution $q(\xi) = a\xi$ in (7.1.5). Then using (7.1.6), we obtain

$$\varphi(x, r) = -ax \left[\cosh^{-1} \frac{x}{\lambda r} - \sqrt{1 - \left(\frac{\lambda r}{x}\right)^2}\right]$$

and velocity components are (7.1.8)

$$u = -a \cosh^{-1} \frac{x}{\lambda r}$$
$$v = a\lambda \sqrt{1 - \left(\frac{x}{\lambda r}\right)^2}.$$

This is a conical solution as all functions depend on x/r only. There is a cone with $x/r = \cot \delta$ where boundary condition is satisfied, i.e. $u/v = \cot \delta$. This gives relation between a and δ

$$a = \frac{V\delta}{\sqrt{\cot^2 \delta - \lambda^2} + \tan \delta \cosh^{-1} \left(\frac{\cot \delta}{\lambda}\right)}$$

For a slender cone $\delta \ll 1$

$$\begin{aligned} & a = V\delta^2 \\ & u = -V\delta^2 \ln \frac{2}{\lambda\delta}, \quad v = V\delta \\ & C_p = 2\delta^2 \left(\ln 2\lambda\delta - \frac{1}{2}\right) \end{aligned} \tag{7.2.4}$$

For plane flows past a wedge, $C_p \sim \delta$, so pressure on a cone has different order of magnitude.

7.3 Similarity rules

Plane flows Remind the equation for potential Potentail of disturbances obeys the equation (6.1.14)

$$\left(1 - M_1^2\right) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{7.3.1}$$

The shape of the boundary may be given in the form

$$y = h_1 Y \left(\frac{x}{L}\right) = \tau_1 L Y \left(\frac{x}{L}\right)$$

with non-dimentional thickness $\tau_1 = h_1/L$, or in comletely non-dimensional form

$$\frac{y}{L} = \tau_1 f \left(\frac{x}{L}\right) \tag{7.3.2}$$

The boundary condition (6.2.3) for φ is

$$\left(\frac{\partial \varphi}{\partial y}\right)_{y=0} = V_1 \frac{dY}{dx} = V_1 \tau_1 Y' \left(\frac{x}{L}\right) \tag{7.3.3}$$

where V_1 is the free-stream velocity.

The pressure coefficient on the boundary is

$$C_{p1} = -\frac{2}{V_1}\left(\frac{\partial \varphi}{\partial x}\right)_{y=0} \tag{7.3.4}$$

Now consider the pontential $\Phi(\xi, \eta)$ pf a second flow. Let Φ be related to φ by the relation

$$\varphi(x, y) = A\frac{V_1}{V_2}\Phi(\xi, \eta) = A\frac{V_1}{V_2}\Phi\left(x, \sqrt{\frac{1-M_1^2}{1-M_2^2}}y\right) \tag{7.3.5}$$

for some constant A. The corrpespondence between coordinate systems is

$$\xi = x, \quad \eta = \sqrt{\frac{1-M_1^2}{1-M_2^2}}y$$

Introducing (7.3.5) in (7.3.1), we find the equation for Φ:

$$\left(1-M_2^2\right)\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} = 0 \tag{7.3.6}$$

Hence, Φ describes a flow with Mach number of M_2. The boundary condition (7.3.3) gives

$$\left(\frac{\partial \varphi}{\partial y}\right)_{y=0} = A\frac{V_1}{V_2}\sqrt{\frac{1-M_1^2}{1-M_2^2}}\left(\frac{\partial \Phi}{\partial \eta}\right)_{y=0} = V_1\tau_1 Y'\left(\frac{x}{L}\right) \tag{7.3.7}$$

The only variable in (7.3.7) is x/L. The equations (7.3.7) can be also written as

$$\left(\frac{\partial \Phi}{\partial \eta}\right)_{\eta=0} = V_2\tau_2 Y'\left(\frac{x}{L}\right)$$

As Y' is the same in both case, we have a relation between A, τ_1 and τ_2:

$$A\sqrt{\frac{1-M_1^2}{1-M_2^2}}\tau_2 = \tau_1 \tag{7.3.8}$$

Using the same function Y means that we consider a family of body shapes. They are not geometrically similar but one shape can be obtained frome another by proper stratching or compression towards the plane $y = 0$.

The prussure coefficients can also be rewrtitten as

$$C_{p1} = -\frac{2}{V_1}\left(\frac{\partial \varphi}{\partial x}\right)_{y=0} = -\frac{2}{V_2}A\left(\frac{\partial \varphi}{\partial \xi}\right)_{\eta=0}$$

For the second flow, the pressure coefficient is

$$C_{p1} = -\frac{2}{V_2}\left(\frac{\partial \varphi}{\partial \xi}\right)_{\eta=0} \qquad (7.3.9)$$

Equations (7.3.8) and (7.3.9) set the similarity rules. Two member of a family of shapes characterized by relative thicknesses τ_1 and τ_2 have the pressure distributions C_{p1} and C_{p2}. If the Mach numbers of the flows are M_1 and M_2, respetively, then $C_{p1} = AC_{p2}$ and

$$\tau_1 = A\sqrt{\frac{1 - M_1^2}{1 - M_2^2}}\tau_2.$$

The same can be expressed by formula

$$\frac{C_p}{A} = g\left(\frac{\tau}{A\sqrt{1 - M^2}}\right) \qquad (7.3.10)$$

g is a function, the factor A is arbitrary.

The crucial point of deriving this similarity rule is linearity of the equation and boundary conditions. The situation is different for transonic flows (nonlinear equations) and axisymmetric flows (nonlinear on the shape function boundary conditions).

Equation (7.3.10) is a generalization of well known similarity rules.

1. If $A = 1$, we have $C_p = g(\tau/\sqrt{1 - M^2})$.
2. If $A = 1/\sqrt{1 - M^2}$, we have $C_p = g(\tau)/\sqrt{1 - M^2}$.
3. If $A = \tau$, we have $C_p = \tau g(\sqrt{1 - M^2})$.
4. If $A = 1/(1 - M^2)$, we have $C_p = g(\tau\sqrt{1 - M^2})/(1 - M^2)$.

The first three methods are different form of Prandtl-Glauert rule. Method 1 states that C_p remains constant if the thickness follows change of Mach number in proper way. Method 2 states that for given shape C_p depends on Mach number as $(1 - M^2)^{-1/2}$, and method 3 states that C_p is proportional to τ for fixed M.

Method 4 is a Goethert rule which is not staightforwards for plane flows but is still valid for axisymmetric ones.

All equations in this subsections were written for subsonic flows, but since only expressions like $\sqrt{(1-M_1^2)/(1-M_2^2)}$ were actually used they are still valid for supersonic ones with change $1-M^2$ to M^2-1. The invariant on the type if flow form of equation (7.3.10) is

$$\frac{C_p}{A} = g_1\left(\frac{\tau^2}{A^2(1-M^2)}\right)$$

Axially symmetric flows For axially symmteric flows it is not possible to state boundary conditions for potential at $r=0$ due to singularity.

Equation for potential is (7.1.1)

$$\left(1-M_1^2\right)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = 0$$

and the transformation between two flows with potential $\varphi(x,r)$ and $\Phi(\xi,R)$is almost the same as for plane case:

$$\varphi(x,r) = A\frac{V_1}{V_2}\Phi(\xi,\eta) = A\frac{V_1}{V_2}\Phi\left(x, \sqrt{\frac{1-M_1^2}{1-M_2^2}}r\right) \tag{7.3.11}$$

The analog to (7.3.3) is

$$\left(\frac{\partial\varphi}{\partial r}\right)_{body} = V_1\tau_1 f'\left(\frac{x}{L}\right)$$

f is the shape function. We cannot move to $r=0$ and have to use exact form

$$\left(\frac{\partial\varphi}{\partial r}\right)_{r=\tau_1 Lf(x/L)} = V_1\tau_1 f'\left(\frac{x}{L}\right)$$

Introducing Φ, we have

$$\left(\frac{\partial\varphi}{\partial r}\right)_{r=\tau_1 Lf(x/L)} = A\frac{V_1}{V_2}\sqrt{\frac{1-M_1^2}{1-M_2^2}}\left(\frac{\partial\Phi}{\partial R}\right)_{R=\tau_1\sqrt{\frac{1-M_1^2}{1-M_2^2}}Lf(x/L)} \tag{7.3.12}$$

On the other hand, Φ is a solution for the problem with incoming velocity of U_2 and the shape function $F(R)$:

$$\left(\frac{\partial\Phi}{\partial R}\right)_{R=\tau_2 LF(x/L)} = V_2\tau_2 F'\left(\frac{x}{L}\right) \tag{7.3.13}$$

In order to compare (7.3.12) and (7.3.12), it is required that the shape functions are the same: $f(x/L) = F(x/L)$, which is the same condition as before. In addition, it is , that

$$\tau_1\sqrt{\frac{1-M_1^2}{1-M_2^2}} = \tau_2.$$

Taking this into account, (7.3.12) gives

$$\tau_1 f'\left(\frac{x}{L}\right) = A\sqrt{\frac{1-M_1^2}{1-M_2^2}}\tau_1\sqrt{\frac{1-M_1^2}{1-M_2^2}}f'\left(\frac{x}{L}\right)$$

This implies the only possible value of A:

$$A = \frac{1-M_2^2}{1-M_1^2} \tag{7.3.14}$$

The pressure coefficient for axially symmetric flows is

$$C_{p1} = -\frac{2}{V_1}\left(\frac{\partial\varphi}{\partial x}\right)_{r=\tau_1 Lf(x/L)} - \frac{1}{V_1^2}\left(\frac{\partial\varphi}{\partial r}\right)^2_{r=\tau_1 Lf(x/L)}$$

Using (7.3.11) we have similar relation in term of Φ:

$$C_{p1} = -\frac{2}{V_2}\left(\frac{\partial\Phi}{\partial\xi}\right)_{R=\tau_1\sqrt{\frac{1-M_1^2}{1-M_2^2}}Lf(x/L)} - \frac{A^2}{V_2^2}\frac{1-M_1^2}{1-M_2^2}\left(\frac{\partial\Phi}{\partial R}\right)^2_{R=\tau_1\sqrt{\frac{1-M_1^2}{1-M_2^2}}Lf(x/L)}$$

Using (7.3.14), we factor out the constant A and obtain

$$\frac{C_p}{A} = g\left(\frac{\tau}{A\sqrt{1-M^2}}\right)$$

Unlike the case of plane flow, A cannot be chosen arbitrary since it must satisfy (7.3.14). This value is $A = (1-M^2)^{-1}$. This gives Goehert's similarity rule:

$$C_p(1-M^2) = g(\tau\sqrt{1-M^2}) \tag{7.3.15}$$

Diving both sides by $\tau^2(1-M^2)$ gives the alternate form

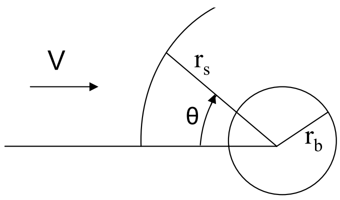
$$\frac{C_p}{\tau^2} = g_1(\tau\sqrt{1-M^2}) \tag{7.3.16}$$

There are no free parameters left in this rule, so it could be adjusted to be valid for transonic flows.

8 Hypersonic flows past a blunt body

8.1 Flow past a sphere

Consider a hypersonic flow of perfect gas with heat capacities ratio γ past a sphere of radius of r_b (pic.13). Let the flow far upstream of the sphere be unifrom, its velocity and density be V and ρ_0.
Assume also $M \gg 1$. The bow shock in front of the sphere is strong near the axis of symetry and the gas density after it is close to the limit one and approximately



Pic. 13: Spherical coordinates

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_0 \equiv \frac{\rho_0}{\varepsilon}, \quad \varepsilon \ll 1$$

Assume that the shape of the bow shock is close the sphere of radius of r_s.

Spherical coordinates r, θ, φ (pic.13) are convenient for this problem. We use Gromeka-Lamb equation of motion taking into account uniformity of the incoming flow

$$\operatorname{curl}\mathbf{V} \times \mathbf{V} = T \operatorname{grad} s \tag{8.1.1}$$

(T is absolute temperature, s is specific entropy).

In our coordinates there is only one non-zero component of $\operatorname{curl}\mathbf{V}$:

$$(\operatorname{curl}\mathbf{V})_\varphi = \frac{1}{r}\left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \vartheta}\right) \tag{8.1.2}$$

Hence, left-hand-side of (8.1.1) takes form

$$\operatorname{curl}\mathbf{V} \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_\varphi \\ 0 & 0 & (rot\,\mathbf{V})_\varphi \\ v_r & v_\theta & 0 \end{vmatrix} \tag{8.1.3}$$

Due to axial symmetry there exists streamfunction ψ:

$$\frac{\partial \psi}{\partial r} = \rho v_\theta r \sin\theta, \quad \frac{\partial \psi}{\partial \theta} = -\rho v_r r^2 \sin\theta \tag{8.1.4}$$

and

$$\begin{aligned}(\operatorname{curl}\mathbf{V})_\varphi &= \frac{1}{r}\left[\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta}\right] = \\ &= \frac{1}{\rho r \sin\theta}\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{\rho r^3}\frac{\cos\theta}{\sin\theta}\frac{\partial \psi}{\partial \theta} + \frac{1}{\rho r^3 \sin\theta}\frac{\partial^2 \psi}{\partial \theta^2}\end{aligned} \tag{8.1.5}$$

Projecting (8.1.3) on $\mathbf{e}_\theta$ direction with (8.1.1) gives

$$v_r\left(\frac{1}{\rho r\sin\theta}\frac{\partial^2\psi}{\partial r^2}-\frac{1}{\rho r^3}\frac{\cos\theta}{\sin\theta}\frac{\partial\psi}{\partial\theta}+\frac{1}{\rho r^3\sin\theta}\frac{\partial^2\psi}{\partial\theta^2}\right)=T\frac{1}{r}\frac{\partial\, s}{\partial\,\theta}\tag{8.1.6}$$

Thus we transform vector equation of motion (8.1.1) to a scalar one. Now, we transform right-hand-side of (8.1.6) with thermodynamic relation

$$dh=Tds+\frac{dp}{\rho}$$

Entropy s depends on ψ only after the shockwave. We can find with dependence since ψ is continuous. In the incoming flow

$$\psi_s=\frac{1}{2}\rho_0 V r_s^2\sin^2\theta\tag{8.1.7}$$

Denote $m=\rho_0 V\cos\theta$ local mass flux and consider Hugoniot adiabat:

$$m^2=\tfrac{p-p_1}{1/\rho_0-1/\rho}\tag{8.1.8}$$

$$\tfrac{\gamma}{\gamma-1}\left(\tfrac{p}{\rho}-\tfrac{p_0}{\rho_0}\right)-\tfrac{p-p_0}{2}\left(\tfrac{1}{\rho}+\tfrac{1}{\rho_0}\right)=0\tag{8.1.9}$$

Equations (8.1.8),(8.1.9) gives expressions for $dp,\ dh$ along the shock wave

$$\begin{aligned}
&dp=2mdm\left(1/\rho_0-1/\rho\right)=-2\rho_0^2V^2\sin\theta\cos\theta\left(1/\rho_0-1/\rho\right)d\theta\\
&dh=\frac{1}{2}dp\left(\frac{1}{\rho}+\frac{1}{\rho_0}\right)\\
&T\,ds=\frac{1}{2}dp\left(\frac{1}{\rho}+\frac{1}{\rho_0}\right)-\frac{dp}{\rho}=\\
&\quad=\rho_1^2V^2\sin\theta\cos\theta\left(1/\rho_1-1/\rho\right)^2d\theta=-V^2\sin\theta\cos\theta(1-\varepsilon)^2d\theta
\end{aligned}\tag{8.1.10}$$

Hence, we can put factor of $\psi_\theta\sim v_r$ explicitly to the right-hand-side of (8.1.6):

$$T\frac{1}{r}\frac{\partial\, s}{\partial\,\vartheta}=T\frac{1}{r}\frac{ds}{d\psi}\frac{\partial\,\psi}{\partial\,\vartheta}\tag{8.1.11}$$

Equation (8.1.10) gives values of s_θ and ϕ_θ right after the shock waves and allows deriving of s_ψ:

$$T\left.\frac{ds}{d\vartheta}\right|_s=T\frac{ds}{d\psi}\left.\frac{\partial\,\psi}{\partial\,\vartheta}\right|_s=-\rho_0V^2\sin\theta\cos\theta(1-\varepsilon)^2$$

and

$$T\frac{ds}{d\psi}=-\frac{V(1-\varepsilon)^2}{\rho_0 r_s^2}\tag{8.1.12}$$

Finally, equations (8.1.4), (8.1.6), (8.1.11),(8.1.12) give equation for ψ

$$\psi_{rr} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\psi_\theta}{\sin\theta}\right) = \frac{Vr^2(1-\varepsilon)^2\sin^2\theta\rho_1}{r_s^2\varepsilon^2} \tag{8.1.13}$$

Boundary conditions for (8.1.13) are continuity of ψ at the shock wave (8.1.7) and tangential velocity $v_\theta = V\sin\theta$ at $r = r_s$.

Equation (8.1.13) has a solution of $\psi = f(r)\sin^2\theta$, with $f(r)$ satisfying ODE

$$f'' - \frac{2}{r^2}f = \frac{\rho V r^2(1-\varepsilon)^2}{\varepsilon^2 r_s^2}$$

Introducing $\xi = r/r_s$ and $g = f/(\rho_0 V r_s^2)$ we obtain Euler equation for $g(\xi)$

$$\xi^2 g'' - 2g = \frac{(1-\varepsilon)^2}{\varepsilon^2}\xi^4 \quad g(1) = \frac{1}{2} \quad g'(1) = \frac{1}{\varepsilon}$$

The solution is

$$g = \frac{1}{3}\left(\frac{1}{2} - 5A + \frac{1}{\varepsilon}\right)\xi + \frac{1}{3}\left(1 + 2A - \frac{1}{\varepsilon}\right)\frac{1}{\xi} + A\xi^4 \quad A = \frac{(1-\varepsilon)^2}{10\varepsilon^2}$$

The position of the sphere corresponds to $g = 0$. At $\xi = 1$ values of g, g', and g'' are known and for $\xi = 1 + \varepsilon\eta$ we have

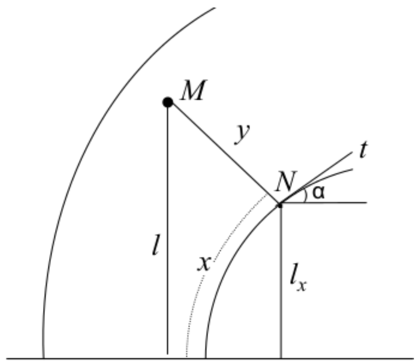
$$g = \frac{1}{2} + \eta + \frac{1}{2}\eta^2$$

which gives $\eta_b = -1$. Returning to physical coordinates results $r_s - r_b = \varepsilon r_s$.

8.2 Basic ideas of Cherny method

For an arbitrary blunt body, there is a thin shock layer between the shock wave and the body surface. If the flow is hypersonic, the density in this layer ρ is much greater than the density in the incoming flow ρ_1. The ration of these densities $\varepsilon = (\gamma-1)/(\gamma+1)$ is small and we can expand all unknown function onto series of ε

Let the body surface is $y = y(x)$, where x is arclength from the axis of symmetry along the tangential surface t, and y is the distance MN from a given point M to the surafce along the normal (14). Let α be the angle between tangent to t and the axis of symmetry, and distances from points N and M are l_t and l. Let us assume that the radius of curvature of the body surface R obeys $|dR/dx| \ll 1$. The streamfunction ψ for these coordinates is



Pic. 14: Curvilinear coordinates

$$d\psi = \rho u l dy - \rho v l(1 + y/R)dx$$

where u and v are velocity componets along $x,\ y$.

As the shock layer is thin, pressure change along x coordinate is much greater than along y. Equation of motion and adiabatic law give

$$\begin{gathered}\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = 0\\ \frac{1}{1+y/R}\frac{\partial v}{\partial x} - \frac{u}{R+y} = -l\frac{\partial p}{\partial \psi};\quad \frac{\partial}{\partial x}\left(\frac{p}{\rho^\gamma}\right) = 0\end{gathered} \tag{8.2.1}$$

Boundary conditions at the shock wave are

$$\begin{aligned}&p = \frac{2}{\gamma+1}\rho_1 V_1^2 \sin^2\beta - \frac{\gamma-1}{\gamma+1}p_1\\ &\frac{\rho_1}{\rho} = \frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\frac{1}{M_1^2\sin^2\beta} =\\ &\quad= \left(\frac{uy'}{1+y/R} - v\right)V_1\left(\cos\alpha - \sin\alpha\frac{y'}{1+y/R}\right)\left[V_1\left(\sin\alpha + \cos\alpha\frac{y'}{1+y/R}\right)\right]^{-1} =\\ &\quad= u + \frac{vy'}{1+y/R}\end{aligned} \tag{8.2.2}$$

where M_1 is Mach number in the incoming flow and β is an angle between tangent to shock wave and the axis of symmetry.

We seek for a solution of (8.2.1) with boundary conditions (8.2.2) in the form

$$\begin{gathered}u = u^{(0)} + \varepsilon u^{(1)} + \ldots;\quad v = \varepsilon v^{(0)} + \ldots\\ p = p^{(0)} + \varepsilon p^{(1)} + \ldots;\quad \rho = \frac{\rho^{(0)}}{\varepsilon} + \rho^{(1)} + \ldots\end{gathered} \tag{8.2.3}$$

The first approximation gives pressure distribution

$$p^{(0)} = \rho_1 V_1^2 \sin^2 \alpha(x) - \frac{1}{R l_t} \int_{\psi}^{\psi^*} u^{(0)} d\psi, \quad \psi^* = \frac{1}{2} \rho_1 V_1 l_t^2$$

Higher order term in (8.2.3) allows finding the thickness of the shock layer.

9 Weak shock wave structure

Gas dynamics usually deals with ideal gas with zero heat conductance. Flows of such medium are isentropic as there are no physical mechanisms for the entropy production. Exceptions usually involve external heat sources or sinks. As a consequence, Euler equations for non-stationary flows are hyperbolic and admit shock waves. These shocks are treated as infinitely thin surface where parameters of the flow change. On the other hand this means that spatial derivatives of those functions are quite large and dissipative process (viscosity and heat conductance) which are proportional to velocity and temperature gradients cannot be neglected. In this section, we take these processes into account and consider the inner structure of a shock wave.

Consider a steady one dimensional flow of a compressible viscous (η is dynamic viscosity coefficient) heat-conductive ($\varkappa$ is heat conductance) gas.

Governing equations are

$$\begin{aligned} &\tfrac{d}{dx}(\rho u) = 0 \\ &\tfrac{d}{dx}\left(p + \rho u - \tfrac{4}{3}\eta\tfrac{du}{dx}\right) = 0 \\ &\rho u\tfrac{dS}{dx} = \tfrac{4}{3}\eta\left(\tfrac{du}{dx}\right)^2 + \tfrac{d}{dx}\left(\varkappa\tfrac{dT}{dx}\right) \end{aligned} \tag{9.0.4}$$

We assume that the second (volume) viscosity coefficient is zero. This assumption is good enough for monoatomic gases and other media when the flow relaxation time is much large than the internal degrees of freedom relaxation time.

The second thermodynamic law $TdS = dh - dp/\rho$ and mass and momentum conservation laws allow rewriting of the entropy equation:

$$\frac{d}{dx}\left[\rho u\left(h + \frac{u^2}{2}\right) - \frac{4}{3}\eta u\frac{du}{dx} - \varkappa\frac{dT}{dx}\right] = 0 \tag{9.0.5}$$

We state boundary conditions for these equations at $-\infty$ and $+\infty$ requiring all parameters ρ, u, p, T be constant. We denote these constants by subindeces 0 and 1, respectively.

First two equations of (9.0.4) and (9.0.5) admit first integrals:

$$\rho u = \rho_0 u_0 \tag{9.0.6}$$

$$p + \rho u^2 - \frac{4}{3}\eta\frac{du}{dx} = p_0 + \rho_0 u_0^2 \tag{9.0.7}$$

$$\rho u\left(h + \frac{u^2}{2}\right) - \frac{4}{3}\eta u\frac{du}{dx} - \varkappa\frac{dT}{dx} = \rho_0 u_0\left(h^{(0)} + \frac{u_0^2}{2}\right) \tag{9.0.8}$$

Considering left-hand-sides at $+\infty$ with zero derivatives, we obtain Rankine-Hugoniot conditions.

Discontinuity is no longer possible, as it means infinite value of derivative du/dx which contradict to (9.0.7). Further we will consider heat conductance and viscosity separately.

9.1 Inviscid heat conductive gas

In this case (9.0.7) is algebraic equation:

$$p + \rho u^2 = p_0 + \rho_0 u_0^2$$

together with (9.0.6) it describes all intermediate states:

$$p = p_0 + \rho_0 u_0^2 \left(1 - \frac{V}{V_0}\right), \quad V = 1/\rho. \tag{9.1.1}$$

At the p, V plane, this equation describes all points on a straight line segment AB which connects initial ("0 point A) and terminal ("1 point B) states at Higoniot adiabat (pic.??). A bit to the left from point A this line is above the Poisson adiabat going through A, the same is valid for points a bit to the right from B. Hence, there is a Poisson adiabat which is tangent to AB. This adiabat corresponds to the maximal value of entropy S_{max}. We can find using the condition of tangent.

As the shock wave is weak $S_1 - S_0$ is of order of $(p_1 - p_0)^3$ or $(V_1 - V_0)^3$. We see that on the segment AB entropy can be larger than S_1 and S_0, so $S - S_0$ can be large than $S_1 - S_0$. That is why we keep the term $S - S_0$ together with $(V - V_0)^2$ in the expansion of $p - p_0$ in a small vicinity of A

$$p - p_0 = \left(\frac{\partial p}{\partial V}\right)_{S_0} (V - V_0) + \frac{1}{2}\left(\frac{\partial^2 p}{\partial V^2}\right)_{S_0} (V - V_0)^2 + \left(\frac{\partial p}{\partial S}\right)_{V_0} (S - S_0)$$

The equation of AB is

$$p - p_0 = \frac{p_1 - p_0}{V_1 - V_0}(V - V_0) = \left(\frac{\partial p}{\partial V}\right)_{S_0} (V - V_0) + \frac{1}{2}\left(\frac{\partial^2 p}{\partial V^2}\right)_{S_0} (V_1 - V_0)(V - V_0)$$

We find the tangent point: it corresponds to $V - V_0 = 1/2(V_1 - V_0)$. The maximal entropy can be easily found:

$$S_{max} - S_0 = \frac{1}{8}\frac{(\partial^2 p/\partial V^2)_{S_0}}{\left(\frac{\partial p}{/}\partial S\right)_{V_0}}(V_1 - V_0)^2.$$

Maximal entropy change in a weak shock wave is a second-order small values. The entropy first rises to its maximal value and then goes down to make the total entropy change be third-order small value. The existence of an extremum of entropy implies the presence of an inflection point on the temperature profile. Indeed, without viscosity, the entropy equation reads

$$\rho u T \frac{dS}{dx} = \varkappa \frac{d^2 T}{dx^2} \tag{9.1.2}$$

We are now able to estimate thickness of the shockwave. We divide both parts of (9.1.2) by T and integrate over x (we take into account that ρu is constant):

$$\rho_0 u_0(S - S_0) = \varkappa \int\limits_{-\infty}^{x} \frac{1}{T}\frac{d^2T}{dx^2}dx = \varkappa \left[\frac{1}{T}\frac{dT}{dx} + \int\limits_{T_0}^{T} \frac{dT}{dx}\frac{1}{T^2}dT \right]. \qquad (9.1.3)$$

For large enough x, we have $dT/dx = 0$ the first term in brackets vanishes and we have

$$\rho_0 u_0(S - S_0) = \varkappa \int\limits_{T_0}^{T} \frac{dT}{dx}\frac{1}{T^2}dT.$$

We define the shock wave thickness Δx for the presence of heat conductance only as

$$\frac{T_1 - T_0}{\Delta x} = \left| \frac{dT}{dx} \right|_{max}. \qquad (9.1.4)$$

Simple estimation gives

$$\rho_0 u_0(S - S_0) = \varkappa \frac{(T_1 - T_0)^2}{dx}\frac{1}{T_0^2}dT.$$

We can replace temperature jump by pressure change:

$$T_1 - T_0 = \left(\frac{\partial T}{\partial p} \right)_S (p_1 - p_0) = \frac{V_0}{c_p}(p_1 - p_0).$$

Here we used the second thermodynamics law formulation: $c_p dT = T ds + pdV$.

In a weak shock wave the entropy change is

$$T_0(S_1 - S_0) = \frac{1}{12}\left(\frac{\partial^2 V}{\partial p^2} \right)_S (p_1 - p_0)^3$$

Using approximate relations: $\partial^2 V \partial p^2 \sim V_0/p_0^2$, $\varkappa \sim \rho_0 c_p l c_0$, $c_0 \sim u_0$, $p_0 \sim c_p \rho_0 T_0$ for mean free path length l, mean chaotic velocity c_0, we obtain

$$\Delta x \sim l \frac{p_0}{p_1 - p_0}$$

From (9.1.3), we see that maximal local entropy change $S_{max} - S_0$ is proportional to $\Delta T/\Delta x \sim (\Delta p)^2$ while terminal entropy change is one order of magnitude less.

In this solution we see that only temperature must be continuous, while all other functions (velcoity, density, pressure) may have discontinuity.

9.2 Viscous non heat conductive gas

Now consider the case $\varkappa = 0$, $\eta > 0$. The entropy equation is

$$\rho u \frac{dS}{dx} = \frac{4}{3}\eta \left(\frac{du}{dx}\right)^2$$

so entropy grows monotonically. The line AB on pV plane is below adiabat going through the point B. The equation of the AB line is

$$p = p_0 + \rho_0 u_0^2 \left(1 - \frac{V}{V_0}\right) + \frac{4}{3}\eta \frac{du}{dx} \tag{9.2.1}$$

As this line is below the straight line AB $du/dx < 0$ inside the shock wave. We define the shock wave thickness as

$$\left|\frac{u_1 - u_0}{\Delta x}\right| = \left|\frac{du}{dx}\right|_{max}.$$

Maximal value of the derivative corresponds to maximal vertical distance between the straight line AB and actual path (9.2.1). Taking a point in the middle between A and B, we estimate

$$\frac{4}{3}\eta \frac{du}{dx} = \frac{1}{8}\left(\frac{\partial^2 p}{\partial V^2}\right)_{S_A} (V_1 - V_0)^2$$

The velocity change is

$$u_0 - u_1 = -\sqrt{(p_1 - p_0)(V_0 - V_1)} = \sqrt{(p_1 - p_0)^2 \left|\frac{\partial V}{\partial p}\right|} \sim \frac{p_1 - p_0}{p_0} c_0$$

and viscosity $\eta \sim \rho_0 l c_0$. The shock wave thickness is

$$\Delta x = l \frac{p_0}{p_1 - p_0}$$

The viscous solution give continuous solution for all functions. This means that friction is a principal mechanism of transferring kinetic energy of the gas into its heat energy.

Considering non-weak shock waves, one may obtain its thickness less than mean free path, which is physically meaningless. The solution of this paradox is that transport coefficients cannot be constant throughout the whole range of temperatures and this fact must be taken into account. For strong waves the solution must be based on kinetics theory and consider possible excitation of internal degrees of freedom.

10 Thermally non-equilibrium flows

10.1 Relaxation to equilibrium

The only way of information transport in gas is collisions between its molecules. Hence there is a reference time scale which is defined by time between collision. It is

$$\tau_{coll} = \frac{l}{c_0} = \frac{1}{nc_0\sigma},$$

where l is mean free path, c_0 — mean velocity of chaotic motion, n is number concentration and σ is collision cross-section. For air at normal conditions l 10^{-8} m and $\tau_{col} \sim 10^{-10}$ s. This time is usually much less than reference time of a gas dynamics problem, so we can assume that collisions take place very often.

On the other hand, there could be situations when one collision is not enough for molecules to exchange energy. This situations occur if quantum effects are sufficient. For diatomic gases two main quantum features may be observed, they connected to internal rotations and oscillations in the molecules.

Energy of the internal rotation is proportional to the moment of inertia and allowed by quantum mechanics angular velocities. The reference temperature $T_r = E_{rot}/k$ for most gases is lower or around 10 K with the exception for H_2 and D_2, these gases have reference rotation temperature around 80 K. Anyway, for room temperature or above gases are in far classical diapason and the energy got of lost by a molecule during a collision is enough to excite or deactivate rotational degrees of freedom. Experiments show that the rotational energy comes to its equilibrium value after about 10 collisions for air gases and 150 —300 collisions for hydrogen and deuterium. This gives an estimation for a relaxation time.

Sometimes, the relaxation to equilibrium takes much longer. Consider this process in details. Let N be a number of molecules which are excited or come to a new state due to chemical reaction, N_e be an equilibrium number of such molecules. In general, there is a law:

$$\frac{dN}{dt} = f(N, T, \rho, \dots)$$

but for small values of relative difference $|N - N_e|/N_e \ll 1$ one can expand this law into Taylor series and obtain

$$\frac{dN}{dt} = \frac{N_e - N}{\tau}$$

with a solution

$$N = N_0 \exp\left(-\frac{t}{\tau}\right) + N_e \left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$$

The value of τ is relaxation time.

If there are several processes with quite different relaxation times one observes series of relaxation according to their timescales.

For a gas dynamics problem, there is another reference time τ_f. For each process with relaxation time τ there are three possibilities:

- $\tau \ll \tau_f$ quasi-equilibrium flow
- $\tau \sim \tau_f$ nonequilibrium flow
- $\tau \gg \tau_f$ frozen flow

In the latter case one can assume that there is no relaxation at all, and the state of non-equilibrium degrees of freedom does not change.

10.2 Sound propagation in a gas with relaxation

Consider small perturbation of the rest state for a gas with relaxation. Let the energy of internal oscillations be the relaxing value. We introduce to temperatures: T corresponding to translational and rotational degrees of freedom and vibrational corresponding to the current level of energy of vibrations. Assume the the gas is perfect with constant heat capacities.

The governing equations are:

$$\begin{array}{lll} \frac{d\rho}{dt} + \rho\nabla\cdot\vec{v} = 0, & \frac{d\vec{v}}{dt} + \frac{1}{\rho}\nabla p & \frac{dh}{dt} - \frac{1}{\rho}\frac{dp}{dt} = 0 \\ h = c_p T + e_v(T_v) & p = \rho R T & \frac{de_v}{dt} = \frac{e_v(T) - e_v(T_v)}{\tau} \end{array} \quad (10.2.1)$$

Excluding h from the third equation (10.2.1) we obtain

$$\frac{dp}{dt} - a_v^2 \frac{d\rho}{dt} + \rho(\gamma - 1)\frac{e_v}{dt} = 0. \quad (10.2.2)$$

Here γ stands for heat capacities ratio for a gas with frozen degree of freedom, hence a_f is called frozen speed of sound.

Consider small perturbations of a uniform state (denoted by primes):

$$\vec{v} = \vec{v}', \quad p = p_0 + p', \quad \rho = \rho_0 + \rho', \quad T = T_0 + T', \quad T_v = T_0 + T_v' \quad (10.2.3)$$

As these perturbations are small with their derivatives, nonlinear term can be neglected after substitution (10.2.3) to (10.2.1), (10.2.2):

$$\begin{array}{cc} \dfrac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v'} = 0, & \dfrac{\partial \vec{v'}}{\partial t} + \dfrac{1}{\rho_0} \nabla p' = 0 \\ \dfrac{\partial p'}{\partial t} - a_f^2 \dfrac{\partial \rho'}{\partial t} + \rho_0(\gamma - 1)\dot{e_v} \dfrac{\partial T_v'}{\partial t} = 0 & \dfrac{\partial T_v}{\partial t} = \dfrac{T' - T_v'}{\tau_0} \\ \dfrac{T'}{T_0} = \dfrac{p'}{p_0} - \dfrac{\rho'}{\rho_0} & \dfrac{de_v}{dt} = \left.\dfrac{de_v}{dT_v}\right|_{T_v = T_0} \end{array} \tag{10.2.4}$$

The second equation implies the existence of velocity potential, so introducing φ:

$$\vec{v'} = \nabla \varphi$$

from two first equations (10.2.4) we have

$$\frac{\partial \rho'}{\partial t} + \rho_0 \Delta \varphi = 0, \quad p' = -\rho_0 \frac{\partial \varphi}{\partial t} \tag{10.2.5}$$

Exclude T' from the last equation:

$$\frac{\partial T_v}{\partial t} = \frac{1}{\tau_0} \left(\frac{T_0}{p_0} p' - \frac{T_0}{rho_0} \rho' - T_v' \right)$$

Taking time derivative of this equation and the third equation (10.2.4 we get

$$\begin{array}{c} \dfrac{\partial^2 p'}{\partial t^2} - a_f^2 \dfrac{\partial^2 \rho'}{\partial t^2} + \rho_0(\gamma - 1)\dot{e_v} \dfrac{\partial^2 T_v'}{\partial t^2} = 0 \\ \dfrac{\partial^2 T_v}{\partial t^2} = \dfrac{1}{\tau_0} \left(\dfrac{T_0}{p_0} \dfrac{\partial p'}{\partial t} - \dfrac{T_0}{rho_0} \dfrac{\partial \rho'}{\partial t} - \dfrac{\partial T_v'}{\partial t} \right) \end{array} \tag{10.2.6}$$

exclude T_v':

$$\frac{\partial^2 p'}{\partial t^2} - a_f^2 \frac{\partial^2 \rho'}{\partial t^2} + \frac{1}{\tau} \left[\left(\lambda \frac{T_0}{p_0} + 1 \right) \frac{\partial p'}{\partial t} - \left(\lambda \frac{T_0}{\rho_0} + a_f^2 \right) \frac{\partial \rho'}{\partial t} \right] = 0$$

where $\lambda = \rho_0(\gamma - 1)\dot{e_v}$, and finally obtain the equation for potential

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \varphi}{\partial t^2} - a_f^2 \Delta \varphi \right) + \frac{1}{\tau} \left(\frac{\partial^2 \varphi}{\partial t^2} - a_e^2 \Delta \varphi \right) = 0. \tag{10.2.7}$$

We use

$$\tau = \tau_0 \left(\frac{\lambda T_0}{p_0} + 1 \right)^{-1}, \quad a_e^2 = \left(\frac{\lambda T_0}{\rho_0} + a_f^2 \right) \left(\frac{\lambda T_0}{p_0} + 1 \right)^{-1}.$$

We simplify these expressions

$$\tau = \tau_0 \left(\frac{\lambda T_0}{p_0} + 1 \right)^{-1} = \frac{\tau_0 p_0}{\rho_0(\gamma - 1)\dot{e_v} T_0 + p_0} = \frac{\tau_0 c_v}{\dot{e_v} + c_v} = \tau_0 \frac{c_v}{c_{ve}}$$

using c_{ve} — constant volume heat capacity for equilibrium state, and

$$a_e^2 = \frac{\lambda T_0 + a_f^2 \rho_0}{\lambda T_0 + p_0} \frac{p_0}{\rho_0} = \frac{\rho_0(\gamma - 1)\dot{e_v} T_0 + \gamma p_0}{\rho_0(\gamma - 1)\dot{e_v} T_0 + p_0} \frac{p_0}{\rho_0} = fraccp + \dot{e_v} c_v + \dot{e_v} \frac{p_0}{\rho_0} = \gamma_e \frac{p_0}{\rho_0}.$$

We see that a_e has the same structure as a_f but it involves heat capacities ration for the equilibrium state. If is clear that $\gamma_e < \gamma$, so $a_e < a_f$.

The equation for small disturbances propagation (10.2.7) involves two wave operators with different speed of sound. Let us investigate its limit cases.

If $\tau \to \infty$ we have a frozen flow and (10.2.7) takes form of usual wave equation with classical (frozen) speed of sound:

$$\frac{\partial^2 \varphi}{\partial t^2} - a_f^2 \Delta \varphi = 0,$$

provided initial conditions satisfies this equation.

It the relaxation time τ is small, we have

$$\frac{\partial^2 \varphi}{\partial t^2} - a_e^2 \Delta \varphi = 0,$$

so disturbances propagate at equilibrium speed of sound.

10.3 Dispersion relation

Consider one dimensional (plane) disturbances. The governing equation (10.2.7) reads

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \varphi}{\partial t^2} - a_f^2 \frac{\partial^2 \varphi}{\partial x^2} \right) + \frac{1}{\tau} \left(\frac{\partial^2 \varphi}{\partial t^2} - a_e^2 \frac{\partial^2 \varphi}{\partial x^2} \right) = 0. \tag{10.3.1}$$

For monochromatic plane wave

$$\varphi = A \exp\left[i(kx - \omega t \right] \tag{10.3.2}$$

it gives dispersion relation

$$-i\omega(-\omega^2 + a_f^2 k^2) + \frac{1}{\tau}(-\omega^2 + a_e^2 k^2) = 0,$$

which is relation between wavenumber and frequency:

$$k = \omega \sqrt{\frac{1 - i\omega\tau}{a_e^2 - i\omega\tau a_f^2}}. \tag{10.3.3}$$

The wavenumber is now complex and $k = k_r + ik_i$, so the disturbance (10.3.2) is rewritten

$$\varphi = A \exp(-k_i x) \exp\left[ik_r\left(x - \frac{\omega}{k_r}t\right)\right]$$

the ratio $\frac{\omega}{k_r}$ is phase velocity and the exponent with real argument $-k_i x$ stands for growth for negative k_i and decay for k_i.

Consider different asymptotic case. Let $\omega\tau \ll 1$. Neglecting terms of order of $(\omega\tau)^2$, we obtain

$$k = \frac{\omega}{a_e} + i\frac{\omega}{a_e}\frac{\omega\tau}{2a_e^2}(a_f^2 - a_e^2).$$

Phase velocity for these waves if equilibrium speed of sound. The imaginary part is positive so disturbances decay and decay rate is proportional to $a_f - a_e > 0$.

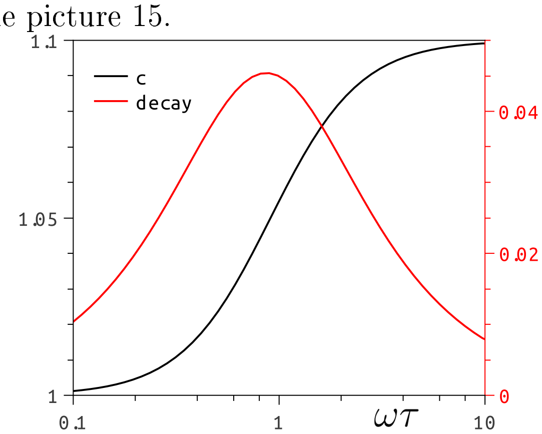
For short (high-frequency waves)

$$k = \frac{\omega}{a_f} + i\frac{1}{2\tau a_f^3}(a_f^2 - a_e^2).$$

For general case the equation (10.3.3) gives

$$k_r = m\omega\cos\theta \qquad k_i = m\omega\sin\theta$$
$$m = \left(\frac{1+(\omega\tau)^2}{a_e^4+(\omega\tau)^2 a_f^4}\right)^{1/4}, \quad 2\theta = \mathrm{atan}\frac{\omega\tau(a_f^2-a_e^2)}{a_e^2+(\omega\tau)^2a_f^2}.$$

The decay rate has maximum at $\omega\tau = a_e/a_f$. Typical dependence of phase velocity ω/k_r and decay rate k_i/ω on normalized frequency $\omega\tau$ is shown at the picture 15.



Pic. 15: Phase velocity and decay rate of disturbances in gas with relaxation

10.4 Flow in a de Laval nozzle

Consider a flow in a de Laval nozzle. For stationary one dimensional flow temperature depends on Mach number as

$$\frac{T_0}{T} = 1 + \frac{\gamma + 1}{2} M^2$$

If a hypersonic flow ($M > 10$) is present at the exit of the nozzle, temperature drops by factor of 40 at the nozzle length. Assuming room temperature near the exit, it is clear that there is very high temperature in the stagnation region, and vibrational degrees of freedom are excited there. Going downstream temperature goes down and these degrees of freedom get deactivated. The energy of molecules or any other thermodynamical parameter corresponding to vibrational degrees of freedom obeys relaxation equation

$$\frac{dq}{dt} = \frac{q_e(\rho, T) - q}{\tau} \tag{10.4.1}$$

for a state parameter q, thermodynamical parameters ρ, T, and relaxation time $\tau(\rho, T)$.
In a nozzle, the flow is one-dimensional and stationary, so $d/dt = u d/dx$, thermodynamical parameters ρ, T are given function of x. Consider some simple solutions.

1. Let $q_e = \text{const}$, $\tau = \text{const}$. The solution of (10.4.1) is

$$q = q_e + (q_0 + q_e)e^{-t/\tau}$$

Considering initial condition $q_0 = q_e$, we have equilibrium flow in the whole nozzle

2 $q_e = q_e(t)$ is a given function, $\tau = \text{const}$. In this case the equation (10.4.1) has a solution

$$q = \frac{1}{\tau} e^{-t/\tau} \int_0^t e^{\xi/\tau} q_e(\xi) d\xi + q_0 e^{-t/\tau}$$

Integrating by parts $(n + 1)$ times, we obtain

$$q = \sum_{i=0}^{n} (-1)^i \tau^i q_e^{(i)}(t) - \left[\sum_{i=0}^{n} (-1)^i \tau^i q_e^{(i)}(0) \right] e^{-t/\tau} + (-1)^{n+1} \tau^n e^{-t/tau} \int_0^t e^{\xi/\tau} q_e^{(n+1)}(\xi) d\xi + q_0 e^{-t/\tau}$$

Assume, there is an equilibrium in the stagnation region. For $n = 2$ we have

$$q = q_e(t) - \tau q_e'(t) + \tau^2 q_e'' - \left[-\tau q_e'(0) + \tau^2 q_e''(0)\right] e^{-t/\tau} - \tau^2 e^{-t/\tau} \int\limits_0^t e^{\xi/tau} q_e(3) d\xi + q_0 e^{-t/\tau}$$

or, if the third derivative is small enough

$$q = q_e(t) \left[1 - \tau \frac{q_e'(t)}{q_e(t)} + \tau^2 \frac{q_e''(t)}{q_e(t)}\right] + A(t) e^{-t/\tau} \qquad (10.4.2)$$

with a bounded function $A(t)$. The reason for non-equilibrium state appearance in the nozzle is a variation of equilibrium along the nozzle.
The reference time of change of q_e is

$$\tau_{var} = \left|\frac{q_e'(t)}{q_e(t)}\right|^{-1}$$

It is the third refence time along with relaxation time τ and the flow refence time $\tau_f = l/u_*$, where l is a refernce length and u_* is reference velocity, e.g. critical speed of sound. The scale τ_{var} only changes along the nozzle, so we consider its local value in different regions.
The governing equation (10.4.1) in non-dimensional form is

$$\frac{dq}{dt} = \frac{q_e(t) - q}{\tau/\tau_f}$$

There are four different cases conserning relation between these scales.

1. $\tau/\tau_f \gg 1$ leads to $dq/dt \approx 0$ and q is almost constant. This is *almost frozen* flow

2. $\tau/\tau_f \ll 1$ implies $q \approx q_e(t)$, so the flow is *almost equilimrium.*

3. τ/τ_{var}. In this case (10.4.2) gives $q \approx q_e(t)$, so the flow is *almost equilimrium.*

4. $\tau \sim \tau_f \sim \tau_{var}$. No simplification is possible and the flow in non-equilibrium.

We assumed that there is an equilibrium at the stagnation region, so in the convergent part of the nozzle it is likely that $\tau/\tau_{var} \ll 1$ and the flow is almost equilibrium.

Further downstream, the velocity of the gas increases and its pressure goes down, so the relaxation time increases. In some region the have non-equilibrium flow as

$$\tau \sim \tau_f \sim \tau_{var}$$

Near the exit area, gas is usually rarefied, the velocity alomost reaches its terminal value, pressure and temperature are quite low. This means that τ_f has its maximum and the relation $\tau_f \sim \tau_{var}$. On the other hand, due to low pressure the relaxation time is small, so

$$\frac{\tau}{\tau_f} \gg 1$$

and the flow is alomost frozen.
One can assume that the middle region is short and neglect changes in this region. In this case there are only equilibrium and frozen flows with a sharp border between them. That is why this method is referred to as 'sudden frrezing method'.
The same procedure can be also applied for a gas jet expansion to vacuum, e.g. from a sonic nozzle. In this case 'sudden freeze' means tranfer from continuum flow to free molecular one.

11 Basics of nucleation theory

The considered above non-equilibruim effects involve a pair of molecules and the equilibruim state of a molecule can be reached during one collision. In this section we consider different process.

11.1 Energy barrier

During isentropic expansion in a nozzle, temperature and presuure of a gas both go down and pressure is power function of temperature. If temperature is low enough, the pressure becomes higher than saturated vapor presuure for this temperature. This means that the gas (vapor) becomes metastable and pahse transition to liquid (condensation) is possible.
The driven force for phase transition is chemical potentials difference:

$$\Delta\mu = \mu^v(p^v, T) - \mu^l(p^v, T) > 0.$$

If a particle goes from vapor phase to a liquid one, the free energy decreases. For saturation conditions $p = p_{sat}$, the pahase are at equilibrium and chenical potential difference is zero.
One can extimate the chemical potential difference:

$$\Delta\mu = [\mu^v(p^v, T) - \mu(p_{sat}, T)] + [\mu(p_{sat}, T) - \mu^l(p^v, T)]$$

The second term describes the chemical potential variation in liquid which is much less than the one for vapor. For thermodynamics, one obtains

$$\Delta\mu = kT \ln S, \quad S = \frac{p^v}{p_{sat}} \tag{11.1.1}$$

introducing supersaturation S.
This equation gives the enegry difference between a particle in bulk vapor and liquid phase. On the other hand, liquid forms clusters with large amount of molecules and thess clusters have a free surface. The formation of surface requires some energy which can be desribed by surface tension coefficient γ. Finally, the energy change for n-cluster formation from n vapor molecules is

$$\Delta G = -nkT \ln S + \gamma A(n)$$

where the surface area $A(n) = 4\pi r_n^2$ and the radius of the cluster r_n is

$$r_n = r^1 n^{1/3}, \quad r^1 = \left(\frac{3v^1}{4\pi}\right)^{1/3}$$

The surface area is

$$A(n) = s_1 n^{2/3} \quad s_1 = (36\pi)^{1/3} \left(v^1\right)^{2/3}$$

Hence,

$$\beta \Delta G = -n \ln S + \theta n^{2/3}, \quad \beta = \frac{1}{kT}, \quad \theta = \frac{\gamma s_1}{kT} \tag{11.1.2}$$

This function has its maximum at

$$n_c = \left(\frac{2}{3}\frac{\theta}{\ln S}\right]^3.$$

The cluster with n_c molecules is called a critical cluster, maximum of the function $\Delta G_* = \Delta G(n_c)$ represents an energy barrier which a system has to overcome to turn to a new (liquid) state. Droplets which are smaller than n_c dissociate on average, while large droplets ($> n_c$) on average grow.
In our approxiamtion about surface energy we obtain classical energy barrier

$$\Delta G_* = \frac{1}{3}\gamma A(n_c) = \frac{16\pi}{3}\frac{\left(v^1\right)^2 \gamma^3}{\left(kT \ln S\right)^2}$$

or in non-dimensional form

$$\beta \Delta G_* = \frac{4}{27}\frac{\theta^3}{\ln^2 S}$$

11.2 Kinetics of nucleation

The equilibrium cluster distribution

$$\rho_e(n) = \rho_1 \exp\left[-\beta \Delta G(n)\right]$$

gives large values for large values of n where the bulk term in ΔG dominates, so this distribution diverges. Hence, the actual distribution is affected by initial conditions and is non-equilibrium.
The classical nucleation theory (CNT) uses the following assumptions:

- the elementary process which changes the size of a nucleus is the attachment to it or loss by it of one molecule
- if a monomer collides a cluster it sticks to it with probability unity
- there is no correlation between successive events that change the number of particles in a cluster (nucleation is a Markov process)

Let f_n be a forwrd rate of attachment of a molecule to an n-cluster (condensation) as a result of which it becomes an $(n+1)$-cluster, and b_n be a backward rate corresponding to loss of a molecule by an n-cluster (evaporation) as a result of which it becomes and $(n1)$-cluster. Then the kinetics of the nucleation process is described by the set of coupled rate equations

$$\frac{\partial\rho(n)}{\partial t} = f_{n-1}\rho(n-1,t) - b_n\rho(n,t) - f_n\rho(n,t) + b_{n+1}\rho(n+1,t)$$

A net rate at which n-clusters become $n+1$ - clusters is

$$J(n,t) = f_n\rho(n,t) - b_{n+1}\rho(n+1,t) \tag{11.2.1}$$

so

$$\frac{\partial rho(n,t)}{\partial t} = J(n-1,t) - J(n,t) \tag{11.2.2}$$

This equiation is called Becker Döring equation.
The coeffcients f_n and b_n are defined independently. The former one is determined by collision frequency nu per unit area:

$$f_n = \nu A(n), \quad \nu = \frac{p^v}{\sqrt{2\pi m_1 kT}}$$

where m_1 is a mass of one molecule.
The backward (evaporation) rate b(n), at which a cluster looses molecules, a priori is not known. It is feasible to assume that this quantity is to a large extent determined by the surface area of the cluster rather than by the properties of the surrounding vapor. Therefore b_n can be assumed to be independent on the actual vapor pressure. In order to find it CNT uses the detailed balance condition at a so called constrained equilibrium state, which would exist for a vapor at the same temperature T and the supersaturation $S > 1$ as the vapor in question. In the constrained equilibrium the net flux is absent $J(n,t) = 0$ since it corresponds to the stage before the nucleation process starts, and the cluster distribution is given by $_e(n)$. From (11.2.1) this implies

$$b_{n+1} = f(n)\frac{\rho_e(n)}{\rho_e(n+1)}$$

and (11.2.1 gives

$$J(n,t)\frac{1}{f(n)\rho_e(n)} = \frac{\rho(n,t)}{\rho_e(n)} - \frac{\rho(n+1,t)}{\rho_e(n+1)} \tag{11.2.3}$$

Assume the nuceation is steady-state, so non-equilibrium concentraions do not depend on time. Hence:

$$J(n,t) = J, \forall n$$

Taking sum of (11.2.3) for n from 1 to some large number N gives

$$J \sum_{n=1}^{N} \left[\frac{1}{f(n)\rho_e(n)} \right] = \frac{\rho(1,t)}{\rho_e(1)} - \frac{\rho(N+1,t)}{\rho_e(N+1)}$$

The first term in right-hand-side is 1 and the second one vanishes as $N \to \infty$. Finally, we get

$$J = \left[\sum_{n=1}^{\infty} \frac{1}{f(n)\rho_e(n)} \right]^{-1}$$

Assume, $n_c \gg 1$ as it is required by assumption of spherical clusters. In this case the sum can be replacd by an integral:

$$J = \left[\int_{n=1}^{\infty} \frac{dn}{f(n)\rho_e(n)} \right]^{-1}$$

The main contribution to the integral comes from n around critical cluster size. In the vicinity of n_c

$$\rho_e \approx \rho_e(n) \exp \left[-\frac{1}{2} \frac{1}{kT} \Delta G''(n_c)(n - n_c)^2 \right], \quad \Delta G''(n_c) < 0$$

The Gaussian intergation gives

$$J = \mathcal{Z} f_{nc} \rho_e(n_c) \quad \mathcal{Z} = \sqrt{-\frac{1}{2\pi} \frac{1}{kT} \Delta G''(n_c)}$$

where $\mathcal{Z}$ is called Zeldovich factor.
Taking into account energy barrier (11.2.2), Zeldovich factor takes the form

$$\mathrm{Z} = \frac{1}{3} \sqrt{\frac{\theta}{\pi}} n_c^{-2/3}$$

or

$$\mathrm{Z} = \sqrt{\frac{\gamma}{kT}} \frac{1}{2\pi \rho_1 r_c^2}$$

Summarizing, the main result of the CNT states that the steady state nucleation rate is an exponential function of the energy barrier. It works well for large enough critical cluster size, which requires small supersaturation.
Nozzle flows can shown τ_f at the region with supersaturation much smaller than the reference time of nucleation. This leads to high supersaturations ($\sim 10^1$ or larger) and CNT is not valid.

Линейная теория устойчивости фронтов горения

Постановка задачи об устойчивости фронтов горения

Исследование устойчивости в линейном приближении проводится путем линеаризации гидродинамических уравнений с учетом граничных условий на поверхности разрыва.

В целях простоты ограничимся рассмотрением случая, когда течение перед фронтом и за ним можно считать несжимаемым .

Пусть невозмущенный фронт находится в плоскости $x = 0$. Скорость u_{01} горючей смеси, занимающей полупространство $x < 0$, направлена в положительную сторону оси x. За разрывом скорость $u_{02} = \tilde{\lambda} u_{01}$ (при медленном горении $\tilde{\lambda} > 1$). Плотность ρ_{01} перед фронтом в $\tilde{\lambda}$ раз больше плотности ρ_{02} позади него, так что $\rho_{01} = \tilde{\lambda}\rho_{02}$.

Рассмотрим отклонения поверхности фронта от плоской. Тогда абсцисса произвольной точки поверхности фронта ζ может быть записана в виде функции от y и t , то есть $\zeta = \zeta(y, t)$.

Соотвествующие искривлению фронта возмущения гидродинамических параметров будут также являться функциями x, y и t. Следовательно, возникает неодномерное нестационарное течение.

В неодномерном случае давление p_1 и компоненты скорости u_1, υ_1 перед фронтом связаны со значениями этих же функций за фронтом p_2, u_2, υ_2 соотношениями

$$
\begin{aligned}
& -\rho_{01}U_1(u_2 - u_1) = (p_2 - p_1)\left[1 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{-1/2}, \\
& -\rho_{01}U_1(\upsilon_2 - \upsilon_1) = -(p_2 - p_1)\frac{\partial\zeta}{\partial y}\left[1 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{-1/2}, \\
& \frac{\partial\zeta}{\partial t}\left[1 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{-1/2} = \left(u_1 - \upsilon_1\frac{\partial\zeta}{\partial y}\right)\left[1 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{-1/2} - U_1\,, \\
& \frac{\partial\zeta}{\partial t}\left[1 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{-1/2} = \left(u_2 - \upsilon_2\frac{\partial\zeta}{\partial y}\right)\left[1 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{-1/2} - U_2\,, \\
& \rho_{01}U_1 = \rho_{02}U_2 = const \quad (4)
\end{aligned}
$$

Здесь U_1 и U_2 есть проекции скоростей газа относительно фронта на нормаль к нему. Первые два из соотношений (4) выражают закон непрерывности потока импульса при переходе через разрыв с учетом того, что в системе координат x, y поверхность фронта не является неподвижной. Два других соотношения - кинематические. Они вытекают из определения скорости поверхности. Последние из соотношений (4) выражают закон сохранения массы и постоянство потока массы смеси и продуктов сгорания.

Пусть в потоке газа возникли малые возмущения скорости и давления

$$u_1 = u_{01} + u_1' \,, \upsilon_1 = \upsilon_1' \,, p_1 = p_{01} + p_1' \,,$$
$$u_2 = u_{02} + u_2' \,, \upsilon_2 = \upsilon_2' \,, p_2 = p_{02} + p_2' \,,$$

Причем

$$u_1' \ll u_{01},\ \upsilon_1' \ll u_{01},\ p_1' \ll p_{01} \,,$$
$$u_2' \ll u_{02},\ \upsilon_2' \ll u_{02},\ p_2' \ll p_{02} \,.$$

В общем случае будет возмущена также поверхность фронта $\zeta = \zeta(y,t)$. Возмущения гидродинамических величин должны удовлетворять уравнениям неразрывности и движения. В смеси получаем линеаризованные уравнения вида

$$\frac{\partial u_1'}{\partial x} + \frac{\partial \upsilon_1'}{\partial y} = 0 \,,$$
$$\frac{\partial u_1'}{\partial t} + u_{01}\frac{\partial u_1'}{\partial x} = -\frac{1}{\rho_{01}}\frac{\partial p_1'}{\partial x} \,,$$
$$\frac{\partial \upsilon_1'}{\partial t} + u_{01}\frac{\partial \upsilon_1'}{\partial x} = -\frac{1}{\rho_{01}}\frac{\partial p_1'}{\partial y} \qquad (5)$$

Три аналогичных уравнения имеют место для газа за фронтом.

Найдем некоторые частные решения уравнений (5). Потребуем, чтобы эти решения были ограничены при $x = \pm\infty$ и удовлетворяли на фронте линеаризованным граничным условиям (4).

Вывод дисперсионного соотношения, критерий устойчивости. Будем искать решение поставленной задачи в форме

$$\frac{u_1'}{u_{01}} = F_1(t)e^{kx} cos\, ky,\ \frac{\upsilon_1'}{u_{01}} = -F_1(t)e^{kx} sin\, ky,$$
$$\frac{p_1'}{\rho_{01}u_{01}^2} = -\left(F_1 + \frac{\tilde{\lambda}}{ku_{02}}F_1'\right)e^{kx} cos\, ky,$$
$$\frac{u_2'}{u_{02}} = \left[F_2(t)e^{-kx} + F_3\left(t - \frac{x}{u_{02}}\right)\right] cos\, ky,$$
$$\frac{\upsilon_2'}{u_{02}} = \left(F_2 e^{-kx} + \frac{F_3'}{ku_{02}}\right) cos\, ky,$$
$$\frac{p_2'}{\rho_{02}u_{02}^2} = -\left(F_2 - \frac{F_2'}{ku_{02}}\right)e^{-kx} cos\, ky,$$
$$\zeta(y,t) = \frac{1}{k}F_4(t) cos\, ky,$$

(k – волновое число).
Легко проверить, что эти выражения удовлетворяют гидродинамическим уравнениям (5). Тогда после алгебраических преобразований получаем уравнение для $F_4(t)$

$$\frac{(\tilde{\lambda}+1)\tilde{\lambda}}{(ku_{02})^2}F_4''+\frac{2\tilde{\lambda}}{ku_{02}}F_4'+ \\ +(1-\tilde{\lambda})F_4=0 \quad (6)$$

Решение уравнения (6) имеет вид $F_4 = F_{40}e^{\delta t}$, где δ определяется дисперсионным соотношением

$$\left(\frac{\delta}{ku_{02}}\right)^2+\frac{2}{1+\tilde{\lambda}}\frac{\delta}{ku_{02}}+ \\ +\frac{1}{\tilde{\lambda}(\tilde{\lambda}+1)}(1-\tilde{\lambda})=0.$$

Из двух корней этого уравнения неустойчивым возмущениям отвечает лишь тот, у которого перед радикалом стоит знак + .
Легко проверить, что неустойчивыми будут возмущения любых длин волн. Обычно $\tilde{\lambda} \gg 1$, так что $\delta \propto \tilde{\lambda}^{-1/2}$.